

Sharp bounds on linear semigroup of Navier Stokes with boundary layer norms

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Abstract

In this paper, we derive sharp bounds on the semigroup of the linearized Navier-Stokes equations near a stationary boundary layer on the half space. The bounds are obtained uniformly in the inviscid limit. Delicate boundary layer norms are introduced in order to capture the true boundary layer behavior of vorticity near the boundary. As an immediate application, we construct an approximate solution which exhibits an L^∞ instability of Prandtl's layers.

Contents

1	Introduction	2
2	Strategy of the proof	6
2.1	Proof of Theorem 1.1	9
3	Elliptic estimates	10
3.1	Boundary layer norms	10
3.2	Inverse of Laplace operator in one space dimension	11
3.3	Stream function and vorticity	12

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4	Semigroup bounds	14
4.1	Bounds on \mathcal{S}_α .	14
4.2	Bounds on \mathcal{R}_α .	22
4.3	Derivative bounds on \mathcal{S}_α .	30
4.4	Derivative bounds on $e^{L_\alpha t}$.	32
5	Construction of an approximate solution	33
5.1	Formal construction	34
5.2	Growing mode	35
5.3	Higher order profiles	35
5.4	Remainder	37
5.5	Proof of Theorem 5.1	37
5.6	Proof of Theorem 1.2	37

1 Introduction

Let L be the linearized Navier Stokes operator around a time independent given shear layer profile $U_s = [U(z), 0]^{tr}$, $z \geq 0$, namely,

$$L\omega = -(U_s \cdot \nabla)\omega - (v \cdot \nabla)\omega_s + \sqrt{\nu}\Delta\omega, \quad (1.1)$$

where $\omega_s = \nabla \times U_s$, $\omega = \nabla \times v$, and $\nabla \cdot v = 0$, with the Dirichlet boundary condition $v = 0$ when $z = 0$. The linearized Navier Stokes equation near U_s , written in term of vorticity, then reads

$$\partial_t \omega - L\omega = 0.$$

We study the linearized problem on the spatial domain $\mathbb{T} \times \mathbb{R}_+$ in the inviscid limit $\nu \rightarrow 0$. Here, in (1.1), $\sqrt{\nu}$ is the inverse of the Reynolds number, computed within a Prandtl's boundary layer of size of order $\sqrt{\nu}$. Throughout the paper, we assume that the boundary layer profile $U(z)$ is real analytic, $U(0) = 0$, and there are positive constants η_0, U_+ so that

$$|\partial_z^k(U(z) - U_+)| \leq C_k e^{-\eta_0 z}, \quad \forall z \geq 0, \quad k \geq 0, \quad (1.2)$$

for some constants C_k .

This is a very classical problem which has led to a huge physical and mathematical literature, focussing in particular on the linear stability, on the dispersion relation, on the study of eigenvalues and eigenmodes, and on the onset of nonlinear instabilities and turbulence; see [1] for an introduction

on these topics, and the classical achievements of Rayleigh, Orr, Sommerfeld, Heisenberg, Tollmien, C.C. Lin, and Schlichting.

In this paper, we are interested in deriving sharp bounds on the semi-group e^{Lt} , with precise boundary layer behaviors, uniformly in the inviscid limit. Precisely, for positive β and γ , let us introduce the one-dimensional boundary layer function space $\mathcal{B}^{\beta,\gamma}$ with its finite norm

$$\|f\|_{\beta,\gamma} = \sup_{z \geq 0} |f(z)| e^{\beta z} \left(1 + \delta^{-1} \phi_P(\delta^{-1} z)\right)^{-1}$$

in which the boundary layer thickness

$$\delta = \gamma \nu^{1/4}$$

and the boundary layer weight function

$$\phi_P(z) = \frac{1}{1 + z^P}$$

for some fixed constant $P > 1$. Here, $\delta = \gamma \nu^{1/4}$ is the thickness of boundary sublayers (as opposed to the main Prandtl's layers) whose appearance is inevitable ([6]).

We expect that the vorticity function $\omega(t, x, z)$, for each fixed t, x , will be in $\mathcal{B}^{\beta,\gamma}$, precisely describing the behavior near the boundary and near infinity. However, the derivative of vorticity will not be in the function space: $\partial_z \omega \notin \mathcal{B}^{\beta,\gamma}$. Therefore, for $p \geq 1$, we are led to introduce the following one-dimensional boundary layer function spaces $\mathcal{B}^{\beta,\gamma,p}$, together with their corresponding finite norms

$$\|f\|_{\beta,\gamma,p} = \sup_{z \geq 0} |f(z)| e^{\beta z} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1} z)\right)^{-1}. \quad (1.3)$$

Note that $\mathcal{B}^{\beta,\gamma,1} = \mathcal{B}^{\beta,\gamma}$. We expect that $\partial_z \omega \in \mathcal{B}^{\beta,\gamma,2}$, and more generally,

$$\partial_z^k \omega \in \mathcal{B}^{\beta,\gamma,1+k}, \quad k \geq 0. \quad (1.4)$$

By convention, $\mathcal{B}^{\beta,\gamma,0} = \mathcal{A}^\beta$, the function space with no boundary layer behavior, which is equipped with the norm $\|f\|_\beta = \sup_{z \geq 0} |f(z)| e^{\beta z}$.

For functions of two variables (x, z) , $f = f(x, z)$, we write

$$f = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} f_\alpha(z).$$

We denote the function space $\mathcal{B}^{\rho,\beta,\gamma,p}$ to be the function space so that $f_\alpha \in \mathcal{B}^{\beta,\gamma,p}$ for all $\alpha \in \mathbb{Z}$ and the norm

$$\|f\|_{\rho,\beta,\gamma,k} := \sum_{\alpha \in \mathbb{Z}} \rho(\alpha) \|f_\alpha\|_{\beta,\gamma,k}$$

is finite, for arbitrarily nonnegative (fixed) weights $\rho(\alpha)$. For simplicity, we take the weight so that $\rho(0) = 0$. Finally, we introduce the following norm for higher derivatives:

$$|||\omega|||_{H_{\text{bl}}^s} = \sum_{a+b \leq s} \|\partial_x^a \partial_z^b \omega\|_{\rho,\beta,\gamma,1+b}. \quad (1.5)$$

Our main result is as follows.

Theorem 1.1. *Let λ_0 be the maximal unstable eigenvalue of the Euler equations (that is, of L with $\nu = 0$) and let $\lambda > \Re \lambda_0$ and $s \geq 0$. We set $\lambda_0 = 0$ if there is no unstable eigenvalue. Then, there is a constant C_λ so that*

$$|||e^{Lt} \omega|||_{H_{\text{bl}}^s} \leq C_\lambda e^{\lambda t} |||\omega|||_{H_{\text{bl}}^s}$$

for any ω with the finite norm $|||\omega|||_{H_{\text{bl}}^s}$.

The main result, Theorem 1.1, is a continuation of [9] to provide sharp and uniform semigroup bounds in the inviscid limit. It also provides a *stable semigroup estimate* for unstable boundary layers. The interest in deriving such a sharp bound on the linearized Navier-Stokes problem is pointed out in [6, 7, 8, 9]. Although the semigroup estimate is a very natural result, but, up to the best of our knowledge, it has never been proven in the literature (except in [9] where we prove it with respect to weighted sup norms without a boundary layer behavior). Its proof relies on a very careful and detailed construction and analysis of the Green function of linear Navier-Stokes equations, constructed in [9], and the Fourier-Laplace approach ([17, 18, 2, 9]).

Let us end the introduction with the following instability result of boundary layers for the Navier-Stokes equations with a source. Precisely, consider

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta + f^\nu(t, x, z) \\ \nabla \cdot u &= 0 \end{aligned} \quad (1.6)$$

on the half plane \mathbb{R}_+^2 , with the Dirichlet boundary condition $u|_{z=0} = 0$. In the inviscid limit $\nu \rightarrow 0$, Prandtl's boundary layers appear. We refer to [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16], among many others, for an extensive study on the classical Prandtl's boundary layers.

Theorem 1.2 (An L^∞ instability of Prandtl's layers). *There exists a smooth shear layer solution*

$$u^\nu(t, y) = U_s(t, \frac{y}{\sqrt{\nu}})$$

of Navier-Stokes equations (1.6) and of the Prandtl's equation, which is unstable in the inviscid limit in the following sense:

- *for arbitrarily large s, p and for arbitrarily small δ , there exist a sequence of solutions v^ν of Navier-Stokes equations (1.6) with sources $f^\nu(t, x, y)$ and a sequence of positive times T^ν such that*

$$\|v^\nu(0) - u^\nu(0)\|_{H^s} + \sup_{t \in [0, T^\nu]} \|f^\nu(t)\|_{H^s} \leq \nu^p,$$

but

$$\|v^\nu(T^\nu) - u^\nu(T^\nu)\|_{L^\infty} \geq \nu^\delta,$$

$$T^\nu \rightarrow 0$$

in the inviscid limit $\nu \rightarrow 0$.

- *for arbitrarily large s, p , there exist a positive constant θ_0 , a sequence of solutions v^ν of Navier-Stokes equations (1.6) with bounded sources $f^\nu(t, x, y)$, and a sequence of positive times T^ν such that*

$$\|v^\nu(0) - u^\nu(0)\|_{H^s} \leq \nu^p,$$

but

$$\|v^\nu(T^\nu) - u^\nu(T^\nu)\|_{L^\infty} \geq \theta_0,$$

$$T^\nu \rightarrow 0$$

in the inviscid limit $\nu \rightarrow 0$.

The paper is outlined as follows. First, we recall in the next section the Fourier-Laplace transform approach and derive the classical Orr-Sommerfeld equations. We then derive elliptic estimates with respect to the boundary layer norms in Section 3. The most technical part of the paper lies in Section 4 where we study carefully the convolution estimates of the Green function for the linear Navier-Stokes problem against boundary layer behavior of vorticity, and finally derive uniform semigroup bounds. Theorem 1.2 will be proved in Section 5 via construction of approximate solutions.

2 Strategy of the proof

The proof of Theorem 1.1 relies on a detailed study of the resolvent solutions. To make it precise, we first observe that the operator L is a compact perturbation of the Laplace operator, and so we can write the semigroup e^{Lt} in term of the resolvent solutions; namely

$$e^{Lt}\omega = \frac{1}{2i\pi} \int_{\Gamma} e^{\lambda t} (\lambda - L)^{-1} \omega d\lambda \quad (2.1)$$

where Γ is a contour on the right of the spectrum of L . We now take the Fourier transform in the x variable, α being the dual discrete Fourier variable, which leads to

$$e^{Lt}\omega = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} e^{L_{\alpha}t} \omega_{\alpha} \quad (2.2)$$

with

$$e^{L_{\alpha}t} \omega_{\alpha} := \frac{1}{2i\pi} \int_{\Gamma} e^{\lambda t} (\lambda - L_{\alpha})^{-1} \omega_{\alpha} d\lambda.$$

In this formula, ω_{α} is the Fourier transform of ω in tangential variables and L_{α} is the Fourier transform of L , which we can now compute explicitly. Indeed, let us introduce the resolvent solution of vorticity

$$\theta_{\alpha} = (\lambda - L_{\alpha})^{-1} \omega_{\alpha}$$

and the corresponding stream function ψ_{α} , defined through the elliptic equation

$$\Delta_{\alpha} \psi_{\alpha} = \theta_{\alpha}.$$

Then the stream function solves

$$\text{OS}(\psi_{\alpha}) := -\varepsilon \Delta_{\alpha}^2 \psi_{\alpha} + (U - c) \Delta_{\alpha} \psi_{\alpha} - U'' \psi_{\alpha} = \frac{\omega_{\alpha}}{i\alpha}, \quad (2.3)$$

$$\psi_{\alpha}|_{z=0} = \partial_z \psi_{\alpha}|_{z=0} = 0, \quad \lim_{z \rightarrow \infty} \psi_{\alpha}(z) = 0, \quad (2.4)$$

where we have denoted

$$\varepsilon = \frac{\sqrt{\nu}}{i\alpha}$$

and

$$c = i\alpha^{-1}\lambda.$$

Note that (2.3) is the classical Orr Sommerfeld equation. Note also that ε is a complex number. As α is an integer and ν goes to 0, ε will be a small

imaginary number. In the sequel we will denote by $\sqrt{\varepsilon}$ the square root of ε which has a positive real part, namely

$$\sqrt{\varepsilon} = \frac{1}{\sqrt{2}}(1 - i)\sqrt{\frac{\sqrt{\nu}}{|\alpha|}}.$$

We then solve the Orr-Sommerfeld equations through its Green function. For each fixed $\alpha \in \mathbb{Z}$ and $c \in \mathbb{C}$, we let $G_{\alpha,c}(x, z)$ be the corresponding Green kernel of the OS problem (2.3)-(2.4). By definition, for each $x \in \mathbb{R}_+$, $G_{\alpha,c}(x, z)$ solves

$$\text{OS}(G_{\alpha,c}(x, \cdot)) = \delta_x(\cdot)$$

on $z \geq 0$, together with the boundary conditions:

$$G_{\alpha,c}(x, 0) = \partial_z G_{\alpha,c}(x, 0) = 0, \quad \lim_{z \rightarrow \infty} G_{\alpha,c}(x, z) = 0.$$

The solution ψ_α to the OS problem (2.3)-(2.4) is then constructed by

$$\psi_\alpha(z) = \int_0^\infty G_{\alpha,c}(x, z) \omega_\alpha(x) \frac{dx}{i\alpha}.$$

Inverting this formula back to the vorticity formulation, we obtain the following lemma.

Lemma 2.1. *Let $G_{\alpha,c}(x, z)$ be the Green function of $\text{OS}(\cdot)$. There hold the integral representation*

$$(\lambda - L_\alpha)^{-1} \omega_\alpha(z) = \frac{1}{i\alpha} \int_0^\infty \Delta_\alpha G_{\alpha,c}(x, z) \omega_\alpha(x) dx \quad (2.5)$$

and

$$e^{L_\alpha t} \omega_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} \int_0^\infty e^{\lambda t} \Delta_\alpha G_{\alpha,c}(x, z) \omega_\alpha(x) \frac{dx d\lambda}{i\alpha}, \quad (2.6)$$

in which $c = i\alpha^{-1}\lambda$ and Γ_α can be chosen, depending on α and lying in the resolvent set of L_α .

It is worth it pointing out that we work with vorticity formulation, which only involves $\Delta_\alpha G_{\alpha,c}(x, z)$, solving

$$\left(-\varepsilon \Delta_\alpha + U - c \right) \Delta_\alpha G_{\alpha,c}(x, z) = \delta_x(z) + U'' G_{\alpha,c}(x, z).$$

This shows that at leading order, the vorticity $\Delta_\alpha G_{\alpha,c}(x, z)$ is governed by the fast dynamics of the operator $-\varepsilon \Delta_\alpha + U - c$. To describe this, we write

$$\Delta_\alpha G_{\alpha,c}(x, z) = \mathcal{G}_a(x, z) + \mathcal{R}_G(x, z) \quad (2.7)$$

in which $\mathcal{G}_a(x, z)$ is the Green function of $-\varepsilon\Delta_\alpha + U - c$ on the whole line, and the residual Green function is computed by

$$\mathcal{R}_G(x, z) = \int_0^\infty \mathcal{G}_a(y, z) U''(y) G_{\alpha, c}(x, y) dy.$$

Roughly speaking, the integration gains an extra small factor, which is precisely the size of the fast oscillation in the Green function $\mathcal{G}_a(x, z)$. Precisely, we recall the following pointwise bounds on the Green function of the Orr-Sommerfeld problem, obtained in [9].

Theorem 2.2 ([9]). *Let $G_{\alpha, c}(x, z)$ be the Green kernel of the Orr-Sommerfeld problem, and write $\Delta_\alpha G_{\alpha, c}(x, z)$ as in (2.7). We set*

$$\mu_s = \alpha, \quad m_f = \inf_{z \geq 0} \hat{\mu}_f(z), \quad M_f = \sup_{z \geq 0} \hat{\mu}_f(z) \quad (2.8)$$

with

$$\hat{\mu}_f(z) = \nu^{-1/4} \sqrt{\lambda + i\alpha U + \alpha^2 \nu}.$$

Then, there hold

$$\mathcal{G}_a(x, z) = \frac{1}{\varepsilon \hat{\mu}_f(x)} e^{-\int_x^z \hat{\mu}_f(y) dy} (1 + \mathcal{O}(\varepsilon)) \quad (2.9)$$

and

$$\begin{aligned} |\partial_x^\ell \partial_z^k \mathcal{R}_G(x, z)| &\leq \frac{C_0 M_f^k}{|\varepsilon m_f^2|} \left(\frac{\mu_s^\ell}{\mu_s} e^{-\theta_0 \mu_s |x-z|} + \frac{M_f^\ell}{m_f} e^{-\theta_0 m_f |x-z|} \right) \\ &\quad + \frac{C_0 M_f^k}{|D(\alpha, c)| |\varepsilon m_f^2|} \left(\frac{\mu_s^\ell}{\mu_s} e^{-\theta_0 \mu_s (|z|+|x|)} + \frac{M_f^\ell}{m_f} e^{-\theta_0 m_f (|z|+|x|)} \right) \end{aligned}$$

for $k, \ell \geq 0$. Here, $D(\alpha, c)$ denotes the Evans function, which vanishes if and only if $\lambda = -i\alpha c$ is the eigenvalue of the linearized Navier-Stokes problem.

In view of the Green function decomposition (2.7) for $\Delta_\alpha G_{\alpha, c}(x, z)$, we write the semigroup

$$e^{L_\alpha t} = \mathcal{S}_\alpha + \mathcal{R}_\alpha \quad (2.10)$$

with

$$\begin{aligned} \mathcal{S}_\alpha \omega_\alpha(z) &:= \frac{1}{2\pi i} \int_{\Gamma_\alpha} \int_0^\infty e^{\lambda t} \mathcal{G}_a(x, z) \omega_\alpha(x) \frac{dx d\lambda}{i\alpha}, \\ \mathcal{R}_\alpha \omega_\alpha(z) &:= \frac{1}{2\pi i} \int_{\Gamma_\alpha} \int_0^\infty e^{\lambda t} \mathcal{R}_G(x, z) \omega_\alpha(x) \frac{dx d\lambda}{i\alpha}. \end{aligned}$$

We then obtain the following key propositions whose proof will be given in the next sections.

Proposition 2.3. *Let $\omega_\alpha \in \mathcal{B}^{\beta,\gamma,p}$ for some positive β so that $\beta \leq 1/4$, and let λ_0 be the maximal unstable eigenvalue of L . Then, for any $p \geq 1$ and $\tau > 0$, there is a positive C_τ so that*

$$\|\mathcal{S}_\alpha \omega_\alpha\|_{\sigma,\beta,\gamma,p} \leq C_\tau e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \|\omega_\alpha\|_{\sigma,\beta,\gamma,p}$$

and

$$\|\mathcal{R}_\alpha \omega_\alpha\|_{\sigma,\beta,\gamma,p} \leq C_\tau e^{(\Re \lambda_0 + \tau)t} \alpha^{-2} \log \alpha \|\omega_\alpha\|_{\sigma,\beta,\gamma,p} \left(1 + \chi_{\{\alpha\delta \leq 1\}} \delta^{1-p} \alpha\right)$$

in which $\chi_{\{\alpha\delta \leq 1\}}$ denotes the usual characteristic function on $\{\alpha\delta \leq 1\}$. Combining the two estimates, we obtain

$$\|e^{L_\alpha t} \omega_\alpha\|_{\sigma,\beta,\gamma,1} \leq C_\tau e^{(\Re \lambda_0 + \tau)t} \|\omega_\alpha\|_{\sigma,\beta,\gamma,1}. \quad (2.11)$$

We also have the following bounds on the derivatives.

Proposition 2.4. *Let $\omega_\alpha \in \mathcal{B}^{\beta,\gamma,p}$ for some positive β so that $\beta \leq 1/4$, and let λ_0 be the maximal unstable eigenvalue of L . Then, for any $p, k \geq 1$ and $\tau > 0$, there is a positive C_τ so that*

$$\|\partial_z^k \mathcal{S}_\alpha \omega_\alpha\|_{\beta,\gamma,p+k} \leq C_\tau e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \sum_{a+b \leq k} \|\alpha^a \partial_z^b \omega_\alpha\|_{\beta,\gamma,p+b}$$

and

$$\|\partial_z^k e^{L_\alpha t} \omega_\alpha\|_{\beta,\gamma,1+k} \leq C_\tau e^{(\Re \lambda_0 + \tau)t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \sum_{a+b \leq k} \|\alpha^a \partial_z^b \omega_\alpha\|_{\beta,\gamma,1+b}.$$

2.1 Proof of Theorem 1.1

Theorem 1.1 follows straightforwardly from the above two propositions. Indeed, we write

$$e^{Lt} w = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} e^{L_\alpha t} \omega_\alpha$$

Hence, it suffices to bound the corresponding Fourier coefficients, which is precisely the content of Proposition 2.4, yielding

$$\|\partial_z^k e^{L_\alpha t} \omega_\alpha\|_{\beta,\gamma,1+k} \leq C_\tau e^{(\Re \lambda_0 + \tau)t} \sum_{a+b \leq k} \|\alpha^a \partial_z^b \omega_\alpha\|_{\beta,\gamma,1+b}.$$

By a view of the norm defined as in (1.5), Theorem 1.1 follows at once. The rest of the paper is devoted to prove Propositions 2.3 and 2.4.

3 Elliptic estimates

In this section, we study in details the boundary layer norms introduced in the Introduction, and provide some elliptic bounds with the boundary layer norms.

3.1 Boundary layer norms

We recall that the one-dimensional boundary layer function spaces $\mathcal{B}^{\beta,\gamma,p}$ are defined by their finite norms

$$\|f\|_{\beta,\gamma,p} = \sup_{z \geq 0} |f(z)| e^{\beta z} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \right)^{-1}.$$

By convention, $\mathcal{B}^{\beta,\gamma,0} = \mathcal{A}^\beta$, the function space with no boundary layer behavior.

Lemma 3.1. *For any functions f and g in $\mathcal{B}^{\beta,\gamma,p}$, the product fg might not be in the same space $\mathcal{B}^{\beta,\gamma,p}$. However, there hold*

$$\|f\|_{\beta,\gamma,p} \leq \|f\|_{\beta,\gamma,q}, \quad (3.1)$$

for $p \geq q \geq 0$, and

$$\|fg\|_{\beta,\gamma,p+p'} \leq \|f\|_{\beta,\gamma,p} \|g\|_{\beta,\gamma,p'}, \quad (3.2)$$

for $p, p' \geq 0$.

Proof. The first inequality is clear. By definition, we compute

$$\begin{aligned} |fg(z)| &\leq \|f\|_{\beta,\gamma,p} \|g\|_{\beta,\gamma,p'} e^{-2\beta z} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \right) \\ &\quad \times \left(1 + \sum_{q=1}^{p'} \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \right). \end{aligned}$$

Since $P > 1$, we can use the inequality

$$\phi_{P+q-1} \phi_{P+q'-1} \leq \phi_{P+q+q'-1}.$$

The lemma follows at once. \square

The next sections are devoted to the study of the classical Laplace operators in these specific analytic spaces.

3.2 Inverse of Laplace operator in one space dimension

Let us now solve the classical Laplace equation

$$\Delta_\alpha \phi = \partial_z^2 \phi - \alpha^2 \phi = f \quad (3.3)$$

on the half line $z \geq 0$, with the Dirichlet boundary condition

$$\phi(0) = 0. \quad (3.4)$$

We start with bounds in the space \mathcal{A}^β of no boundary layers. We will prove

Proposition 3.2. *If $f \in \mathcal{A}^\beta$, then $\phi \in \mathcal{A}^\beta$ provided $\beta < 1/2$. In addition, there holds*

$$\alpha^2 \|\phi\|_\beta + |\alpha| \|\partial_z \phi\|_\beta + \|\partial_z^2 \phi\|_\beta \leq C \|f\|_\beta, \quad (3.5)$$

where the constant C is independent of the integer $\alpha \neq 0$.

Note that the multiplication by α can be seen as a derivative in x variable, using Fourier transform, which explains the orders of α appearing in (3.5). As expected, by inverting the Laplace operator in these spaces we gain control on two derivatives in both variables.

Proof. The solution ϕ of (3.3) is explicitly given by

$$\phi(z) = -\frac{1}{2\alpha} \int_0^\infty \left(e^{-\alpha|x-z|} - e^{-\alpha|x+z|} \right) f(x) dx. \quad (3.6)$$

A direct bound leads to

$$\|\phi\|_\beta \leq \frac{C}{\alpha^2} \|f\|_\beta$$

in which the extra factor of α^{-1} was due to the x -integration. Differentiating the integral formula, we get

$$\|\partial_z \phi\|_\beta \leq \frac{C}{\alpha} \|f\|_\beta.$$

We then use the equation to bound $\partial_z^2 \phi$, which ends the proof. \square

We now establish a similar property for $\mathcal{B}^{\beta,\gamma}$ norms:

Proposition 3.3. *If $f \in \mathcal{B}^{\beta,\gamma}$, then $\phi \in \mathcal{A}^\beta$ provided $\beta < 1/2$. In addition, there holds*

$$|\alpha| \|\phi\|_{\beta,0} + \|\partial_z \phi\|_{\beta,0} \leq C \|f\|_{\beta,\gamma}, \quad (3.7)$$

where the constant C is independent of α .

Proof. We will only consider the case $\alpha > 0$, the opposite case being similar. The Green function of $\partial_z^2 - \alpha^2$ is

$$G(x, z) = \frac{1}{\alpha} \left(e^{-\alpha|z-x|} - e^{-\alpha|z+x|} \right)$$

and is bounded by α^{-1} . Therefore, using (3.6),

$$\begin{aligned} |\phi(z)| &\leq \alpha^{-1} \|f\|_{\beta, \gamma} \int_0^\infty e^{-\alpha|z-x|} e^{-\beta x} \left(1 + \delta^{-1} \phi_P(\delta^{-1}x) \right) dx \\ &\leq \alpha^{-1} \|f\|_{\beta, \gamma} \left(\alpha^{-1} + \delta^{-1} \int_0^\infty \phi_P(\delta^{-1}x) dx \right) \end{aligned}$$

which yields the claimed bound for ϕ since $P > 1$. A similar proof applies for $\partial_z \phi$. \square

Note that this Proposition only gives bounds on first order derivatives of ϕ . As the source term f has a boundary layer behavior we cannot get a good control on second order derivatives. To get bounds on second order derivatives we need to use an extra control on f . For instance, as a direct consequence of the previous proposition, we have, for nonzero integers α ,

$$\alpha^2 \|\phi\|_{\beta, 0} + |\alpha| \|\partial_z \phi\|_{\beta, 0} + \|\partial_z^2 \phi\|_{\beta, \gamma} \leq C \|\alpha f\|_{\beta, \gamma}. \quad (3.8)$$

Note that the bound on $\partial_z^2 \phi$ is recovered using directly $\partial_z^2 \phi = \alpha^2 \phi + f$.

3.3 Stream function and vorticity

Let us now turn to the two dimensional Laplace operator.

Proposition 3.4. *Let ϕ be the solution of*

$$-\Delta \phi = \omega$$

with the zero Dirichlet boundary condition, and let

$$v = \nabla^\perp \phi.$$

If $\omega \in \mathcal{B}^{\rho, \beta, \gamma}$, then $\phi \in \mathcal{B}^{\rho, \beta, 0}$ and $v = (v_1, v_2) \in \mathcal{B}^{\rho, \beta, 0}$. Moreover, there hold the following elliptic estimates

$$\|\phi\|_{\rho, \beta, 0} + \|v_1\|_{\rho, \beta, 0} + \|v_2\|_{\rho, \beta, 0} \leq C \|\omega\|_{\rho, \beta, \gamma}, \quad (3.9)$$

$$\begin{aligned} \|\partial_x v_1\|_{\rho, \beta, 0} + \|\partial_x v_2\|_{\rho, \beta, 0} + \|\partial_z v_1\|_{\rho, \beta, 1} + \|\partial_z v_2\|_{\rho, \beta, 0} \\ \leq C \|\omega\|_{\rho, \beta, \gamma} + C \|\partial_x \omega\|_{\rho, \beta, \gamma}, \end{aligned} \quad (3.10)$$

and, with $\psi(z) = \frac{z}{1+z}$,

$$\|\psi^{-1}v_2\|_{\rho,\beta,0} \leq C\|\omega\|_{\rho,\beta,\gamma} + C\|\partial_x\omega\|_{\rho,\beta,\gamma}, \quad (3.11)$$

for some constant C .

Note that, as previously, because of the boundary layer behavior of ω , we need to add a control on one derivative of ω , namely on $\partial_x\omega$, to get a full control on the derivatives of the velocity. Once again we only "gain" one derivative.

Proof. The proof relies on the Fourier transform in the x variable, with dual integer Fourier component α . We then have

$$\partial_z^2\phi_\alpha - \alpha^2\phi_\alpha = -f_\alpha.$$

Bounds (3.9) is then a direct consequence of Proposition 3.3. Bound (3.10) is a consequence of (3.8), and (3.11) comes from the integration of $\partial_z v_2$ together with (3.10). \square

Next, let us study quadratic nonlinear terms of the type $u \cdot \nabla \tilde{\omega}$. We have the following lemma.

Lemma 3.5. *For all $s > 1$, there holds*

$$\|(u \cdot \nabla)\tilde{\omega}\|_{H_{\text{bl}}^s} \leq C\|\omega\|_{H_{\text{bl}}^{s+1}}\|\tilde{\omega}\|_{H_{\text{bl}}^s} + C\|\omega\|_{H_{\text{bl}}^s}\|\tilde{\omega}\|_{H_{\text{bl}}^{s+1}} \quad (3.12)$$

with $u = \nabla^\perp \Delta^{-1}\omega$, for two different vorticity ω and $\tilde{\omega}$. Here, we recall the boundary layer Sobolev norms

$$\|\omega\|_{H_{\text{bl}}^s} = \sum_{a+b \leq s} \|\partial_x^a \partial_z^b \omega\|_{\rho,\beta,\gamma,1+b}. \quad (3.13)$$

Proof. We write

$$(u \cdot \nabla)\tilde{\omega} = u_1 \partial_x \tilde{\omega} + u_2 \partial_z \tilde{\omega}.$$

By definition, we compute

$$\begin{aligned} \|u_1 \partial_x \tilde{\omega}\|_{H_{\text{bl}}^s} &= \sum_{a+b \leq s} \|\partial_x^a \partial_z^b (u_1 \partial_x \tilde{\omega})\|_{\rho,\beta,\gamma,1+b} \\ &= \sum_{a+b \leq s} \sum_{b_1+b_2=b} \sum_{a_1+a_2=a} \|\partial_x^{a_1} \partial_z^{b_1} u_1 \partial_x^{1+a_2} \partial_z^{b_2} \tilde{\omega}\|_{\rho,\beta,\gamma,1+b_1+b_2} \\ &\leq \sum_{a+b \leq s} \sum_{b_1+b_2=b} \sum_{a_1+a_2=a} \|\partial_x^{a_1} \partial_z^{b_1} u_1\|_{\rho,\beta,\gamma,b_1} \|\partial_x^{1+a_2} \partial_z^{b_2} \tilde{\omega}\|_{\rho,\beta,\gamma,1+b_2} \end{aligned}$$

in which the last inequality uses the algebra structure (3.2) of the boundary layer norms. Using the elliptic estimates from Proposition 3.4, we obtain at once

$$\|u_1 \partial_x \tilde{\omega}\|_{H_{\text{bl}}^s} \leq C \|\omega\|_{H_{\text{bl}}^s} \|\tilde{\omega}\|_{H_{\text{bl}}^{s+1}}$$

Similarly, we write

$$\|u_2 \partial_z \tilde{\omega}\|_{H_{\text{bl}}^s} = \sum_{a+b \leq s} \sum_{b_1+b_2=b} \sum_{a_1+a_2=a} \|\partial_x^{a_1} \partial_z^{b_1} u_2 \partial_x^{a_2} \partial_z^{1+b_2} \tilde{\omega}\|_{\rho, \beta, \gamma, 1+b_1+b_2}$$

in which for $b_1 \neq 0$, we have

$$\begin{aligned} \|\partial_x^{a_1} \partial_z^{b_1} u_2 \partial_x^{a_2} \partial_z^{1+b_2} \tilde{\omega}\|_{\rho, \beta, \gamma, 1+b_1+b_2} &\leq \|\partial_x^{a_1} \partial_z^{b_1} u_2\|_{\rho, \beta, \gamma, b_1-1} \|\partial_x^{a_2} \partial_z^{1+b_2} \tilde{\omega}\|_{\rho, \beta, \gamma, 2+b_2} \\ &\leq C \|\omega\|_{H_{\text{bl}}^s} \|\tilde{\omega}\|_{H_{\text{bl}}^{s+1}}. \end{aligned}$$

It remains to bound the term $\partial_x^{a_1} u_2 \partial_x^{a_2} \partial_z^{1+b} \tilde{\omega}$. Using the fact that u_2 vanishes on the boundary $z = 0$ and setting $\psi(z) = \frac{z}{1+z}$, we have

$$\begin{aligned} \|\partial_x^{a_1} u_2 \partial_x^{a_2} \partial_z^{1+b} \tilde{\omega}\|_{\rho, \beta, \gamma, 1+b} &\leq \|\psi(z)^{-1} \partial_x^{a_1} u_2\|_{\rho, \beta, \gamma, 0} \|\psi(z) \partial_x^{a_2} \partial_z^{1+b} \tilde{\omega}\|_{\rho, \beta, \gamma, 1+b} \\ &\leq \|\psi(z)^{-1} \partial_x^{a_1} u_2\|_{\rho, \beta, \gamma, 0} \|\partial_x^{a_2} \partial_z^{1+b} \tilde{\omega}\|_{\rho, \beta, \gamma, 2+b} \\ &\leq C \|\omega\|_{H_{\text{bl}}^{s+1}} \|\tilde{\omega}\|_{H_{\text{bl}}^s} + C \|\omega\|_{H_{\text{bl}}^s} \|\tilde{\omega}\|_{H_{\text{bl}}^{s+1}}. \end{aligned}$$

The lemma follows. \square

4 Semigroup bounds

4.1 Bounds on \mathcal{S}_α .

In this section, we prove the bounds on \mathcal{S}_α . By the explicit construction of the Green function, $\mathcal{G}_a(x, z)$ is holomorphic in λ , except on the complex half strip:

$$\mathcal{H}_\alpha(x, z) := \left\{ \lambda = -k - \alpha^2 \sqrt{\nu} + i\alpha U(y), \quad k \in \mathbb{R}_+, \quad y \in [x, z] \right\}.$$

In what follows, we shall use the Cauchy's theory to decompose the contour Γ_α of integration in the complex plane, so that Γ_α remains outside this complex strip. We note that the eigenvalues $\hat{\mu}_f(x) = \nu^{-1/4} \sqrt{\lambda + i\alpha U(x) + \alpha^2 \sqrt{\nu}}$ changes its sign when crossing the half line $\lambda = -k - \alpha^2 \sqrt{\nu} + i\alpha U(x)$, with $k \in \mathbb{R}_+$.

Let us recall

$$\mathcal{G}_a(x, z) = \frac{1}{\varepsilon \hat{\mu}_f(x)} e^{-\int_x^z \hat{\mu}_f(y) dy} (1 + \mathcal{O}(\varepsilon)).$$

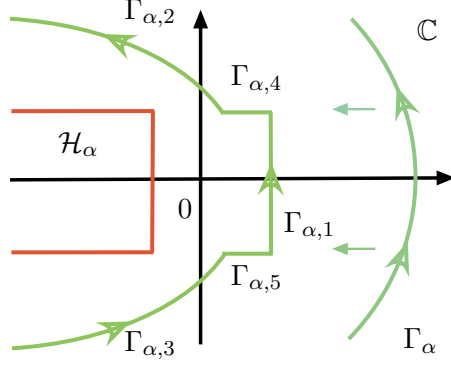


Figure 1: Shown the decomposition of contour Γ_α of integration.

We first consider the case when $x < z$. We set

$$\begin{aligned}\Gamma_{\alpha,1} &:= \left\{ \lambda = \gamma - \alpha^2 \sqrt{\nu} - i\alpha c, \quad \min_{y \in [x,z]} U(y) \leq c \leq \max_{y \in [x,z]} U(y) \right\} \\ \Gamma_{\alpha,2} &:= \left\{ \lambda = (a^2 - k^2 - \frac{1}{2}\alpha^2)\sqrt{\nu} - i\alpha \min_{[x,z]} U + 2\sqrt{\nu}iak, \quad k \geq 0 \right\} \\ \Gamma_{\alpha,3} &:= \left\{ \lambda = (a^2 - k^2 - \frac{1}{2}\alpha^2)\sqrt{\nu} - i\alpha \max_{[x,z]} U + 2\sqrt{\nu}iak, \quad k \leq 0 \right\}\end{aligned}$$

in which we take

$$a = \frac{|x - z|}{2\sqrt{\nu t}}. \quad (4.1)$$

See Figure 1. The choice of the parabolic contours $\Gamma_{\alpha,2}$ and $\Gamma_{\alpha,3}$ is necessary to avoid singularities in small time ([19, 9]). We stress that they never meet the complex strip $\mathcal{H}_\alpha(x, z)$.

We fix an arbitrarily small, but fixed, positive constant θ_0 . We consider two following cases.

- Case 1: $a^2 \sqrt{\nu} \geq \theta_0$. In this case, we take

$$\gamma := a^2 \sqrt{\nu} + \frac{1}{2} \alpha^2 \sqrt{\nu} \quad (4.2)$$

and the contour of integration is taken to be

$$\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,2} \cup \Gamma_{\alpha,3}.$$

- Case 2: $a^2 \sqrt{\nu} \leq \theta_0$. In this case,

$$\gamma := \theta_0 + \frac{1}{2} \alpha^2 \sqrt{\nu} \quad (4.3)$$

and the contour of integration is taken to be

$$\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,2} \cup \Gamma_{\alpha,3} \cup \Gamma_{\alpha,4} \cup \Gamma_{\alpha,5}$$

in which

$$\begin{aligned}\Gamma_{\alpha,4} &:= \left\{ \lambda = k - \frac{1}{2}\alpha^2\sqrt{\nu} - i\alpha \min_{[x,z]} U, \quad a^2\sqrt{\nu} \leq k \leq \theta_0 \right\} \\ \Gamma_{\alpha,5} &:= \left\{ \lambda = k - \frac{1}{2}\alpha^2\sqrt{\nu} - i\alpha \max_{[x,z]} U, \quad a^2\sqrt{\nu} \leq k \leq \theta_0 \right\}.\end{aligned}$$

We stress that in both cases, we have

$$\gamma \geq \theta_0 + \frac{1}{2}\alpha^2\sqrt{\nu}. \quad (4.4)$$

Bounds on $\Gamma_{\alpha,1}$.

We start our computation with the integral on $\Gamma_{\alpha,1}$. Note that $\sqrt{a+ib} \geq \sqrt{a}$ for any real numbers a, b . Thus, for $\lambda \in \Gamma_{\alpha,1}$ and $y \in [x, z]$, we have

$$\Re \hat{\mu}_f(y) = \nu^{-1/4} \Re \sqrt{\gamma + i\alpha(U(y) - c)} \geq \nu^{-1/4} \sqrt{\gamma}$$

and

$$|\varepsilon \hat{\mu}_f(x)| = \nu^{1/4} \alpha^{-1} |\sqrt{\gamma + i\alpha(U(x) - c)}|.$$

We compute

$$\begin{aligned}& \left| \int_{\Gamma_{\alpha,1}} e^{\lambda t} \mathcal{G}_a(x, z) \frac{d\lambda}{i\alpha} \right| \\ & \leq e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} \int_{\Gamma_{\alpha,1}} \frac{1}{|\varepsilon \hat{\mu}_f(x)|} e^{-\int_x^z \hat{\mu}_f(y) dy} \left(1 + \mathcal{O}(\varepsilon)\right) \frac{|d\lambda|}{\alpha} \\ & \leq e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\nu^{-1/4} \sqrt{\gamma} |x-z|} \int_{\min_{[x,z]} U}^{\max_{[x,z]} U} \frac{\alpha dc}{\nu^{1/4} |\sqrt{\gamma + i\alpha(U(x) - c)}|} \\ & \leq C_0 e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\nu^{-1/4} \sqrt{\gamma} |x-z|} \nu^{-1/4} |\sqrt{\gamma + i\alpha(U(x) - c)}| \Big|_{\min_{[x,z]} U}^{\max_{[x,z]} U} \\ & \leq C_0 e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\nu^{-1/4} \sqrt{\gamma} |x-z|} \nu^{-1/4} \sqrt{\gamma + \alpha|x-z| \|U'\|_{L^\infty}} \\ & \leq C_0 e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\nu^{-1/4} \sqrt{\gamma} |x-z|} \nu^{-1/4} \sqrt{\gamma} \sqrt{1 + \gamma^{-1} \alpha|x-z| \|U'\|_{L^\infty}}.\end{aligned}$$

We first take care of $e^{\gamma t}$. In the case that γ is defined as in (4.3), we have

$$e^{\gamma t} e^{-\alpha^2 \nu t} e^{-\nu^{-1/2} \sqrt{\gamma} |x-z|} \leq e^{\theta_0 t} e^{-\frac{1}{2} \alpha^2 \nu t} e^{-\nu^{-1/2} \sqrt{\gamma} |x-z|}.$$

When γ is defined by (4.2), using $\sqrt{\gamma} \geq a\nu^{1/4}$ and the definition of a , we compute

$$\begin{aligned} e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\nu^{-1/4} \sqrt{\gamma} |x-z|} &\leq e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\frac{1}{2} a |x-z|} e^{-\frac{1}{2} \nu^{-1/4} \sqrt{\gamma} |x-z|} \\ &\leq e^{-\frac{1}{2} \alpha^2 \sqrt{\nu} t} e^{-\frac{1}{2} \nu^{-1/4} \sqrt{\gamma} |x-z|}. \end{aligned}$$

In addition, using the fact that $\gamma \geq \theta_0 > 0$, $\sqrt{\gamma} \geq \frac{1}{2} \alpha \nu^{-1/4}$, and the inequality $\sqrt{1+X} e^{-X} \leq C_0 e^{-X/2}$, we estimate

$$e^{-\nu^{-1/4} \sqrt{\gamma} |x-z|} \leq e^{-\frac{1}{2} \alpha |x-z|} e^{-\frac{1}{2} \nu^{-1/4} \sqrt{\gamma} |x-z|}$$

and

$$e^{-\frac{1}{2} \alpha |x-z|} \sqrt{1 + \gamma^{-1} \alpha |x-z| \|U'\|_{L^\infty}} \leq C_0.$$

Putting these into the above estimate, we have obtained

$$\left| \int_{\Gamma_{\alpha,1}} e^{\lambda t} \mathcal{G}_a(x, z) \frac{d\lambda}{i\alpha} \right| \leq C_0 e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} e^{-\frac{1}{2} \nu^{-1/4} \sqrt{\gamma} |x-z|} \sqrt{\gamma} \nu^{-1/4}.$$

Clearly, the same bound holds for $x \geq z$.

We now study the convolution with the boundary layer data $\omega_\alpha(z)$, satisfying

$$|\omega_\alpha(z)| \leq \|\omega_\alpha\|_{\beta, \gamma, p} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1} z) \right) e^{-\beta z}. \quad (4.5)$$

For sake of presentation, we can of course assume that $\|\omega_\alpha\|_{\beta, \gamma, p} = 1$. We compute

$$\begin{aligned} &\left| \int_0^\infty \int_{\Gamma_{\alpha,1}} e^{\lambda t} \mathcal{G}_a(x, z) \omega_\alpha(x) \frac{d\lambda}{i\alpha} dx \right| \\ &\leq C_0 e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t} \nu^{-1/4} \sqrt{\gamma} \int_0^\infty e^{-\frac{1}{2} \nu^{-1/4} \sqrt{\gamma} |x-z|} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1} x) \right) e^{-\beta x} dx. \end{aligned}$$

We shall estimate term by term. First, using the triangle inequality $|x| \geq |z| - |x-z|$ and the fact that for $\beta \leq \frac{1}{4} \nu^{-1/4} \sqrt{\gamma}$, we note that

$$e^{-\frac{1}{2} \nu^{-1/4} \sqrt{\gamma} |x-z|} e^{-\beta |x|} \leq e^{-\beta |z|} e^{-\frac{1}{4} \nu^{-1/4} \sqrt{\gamma} |x-z|}.$$

This yields the spatial decay after the integration. In addition, the integral without the boundary layer behavior is straightforward:

$$\nu^{-1/4} \sqrt{\gamma} \int_0^\infty e^{-\frac{1}{4} \nu^{-1/4} \sqrt{\gamma} |x-z|} \leq C_0. \quad (4.6)$$

As for the boundary layer behavior, for each $1 \leq q \leq p$, we write

$$\begin{aligned} & \nu^{-1/4} \sqrt{\gamma} \int_0^\infty e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ &= \nu^{-1/4} \sqrt{\gamma} \left(\int_{z/2}^\infty + \int_0^{z/2} \right) e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx. \end{aligned}$$

Since the boundary layer weight $\phi_{P-1+q}(\cdot)$ is decreasing, we have

$$\begin{aligned} & \nu^{-1/4} \sqrt{\gamma} \int_{z/2}^\infty e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ & \leq C_0 \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \nu^{-1/4} \sqrt{\gamma} \int_0^\infty e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} dx \\ & \leq C_0 \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \end{aligned}$$

by a view of (4.6). Now as for the integral over $(0, \frac{z}{2})$, we note that for $0 \leq x \leq \frac{z}{2}$, we have $|x-z| \geq \frac{1}{2}|z|$ and so

$$e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} \leq e^{-\frac{1}{16}\nu^{-1/4}\sqrt{\gamma}|z|} e^{-\frac{1}{8}\nu^{-1/4}\sqrt{\gamma}|x-z|}.$$

Hence, since the boundary layer weight is bounded by 1, we have

$$\begin{aligned} & \nu^{-1/4} \sqrt{\gamma} \int_0^{z/2} e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ & \leq \delta^{-q} \nu^{-1/4} \sqrt{\gamma} \int_0^{z/2} e^{-\frac{1}{16}\nu^{-1/4}\sqrt{\gamma}|z|} e^{-\frac{1}{8}\nu^{-1/4}\sqrt{\gamma}|x-z|} dx \\ & \leq C_0 \delta^{-q} e^{-\frac{1}{16}\nu^{-1/4}\sqrt{\gamma}|z|}. \end{aligned}$$

Now we recall that the boundary layer thickness $\delta = \nu^{1/4}$, whereas the boundary layer behavior coming from the Green function: $e^{-\frac{1}{16}\nu^{-1/4}\sqrt{\gamma}|z|}$ is of smaller thickness, since $\sqrt{\gamma} \geq \sqrt{\theta_0}$. Hence,

$$e^{-\frac{1}{16}\nu^{-1/4}\sqrt{\gamma}|z|} \leq C_0 \phi_{P-1+q}(\delta^{-1}z).$$

Combining the above estimates, we obtain

$$\nu^{-1/4} \sqrt{\gamma} \int_0^\infty e^{-\frac{1}{4}\nu^{-1/4}\sqrt{\gamma}|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \leq C_0 \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \quad (4.7)$$

for all $1 \leq q \leq p$. Summing up in q , together with (4.6), we obtain

$$\left\| \int_0^\infty \int_{\Gamma_{\alpha,1}} e^{\lambda t} \mathcal{G}_a(x, \cdot) \omega_\alpha(x) \frac{d\lambda}{i\alpha} dx \right\|_{\beta, \gamma, p} \leq C_0 e^{\gamma t} e^{-\alpha^2 \sqrt{\nu} t}. \quad (4.8)$$

Bounds on $\Gamma_{\alpha,2}$ and $\Gamma_{\alpha,3}$.

By symmetry, it suffices to give bounds on $\Gamma_{\alpha,2}$. For $\lambda \in \Gamma_{\alpha,2}$ and $y \in [x, z]$, we compute

$$\begin{aligned}\hat{\mu}_f(y) &= \nu^{-1/4} \sqrt{(a^2 - k^2)\sqrt{\nu} + i\alpha(U - \min_{[x,z]} U) + 2i\sqrt{\nu}ak} \\ &\geq \nu^{-1/4} \sqrt{(a^2 - k^2)\sqrt{\nu} + 2i\sqrt{\nu}ak} \\ &= \sqrt{(a + ik)^2} = a.\end{aligned}$$

So, using $a = \frac{|x-z|}{2\sqrt{\nu}t}$, we get

$$\begin{aligned}e^{\lambda t} e^{-\int_x^z \hat{\mu}_f(y) dy} &\leq e^{a^2 \sqrt{\nu}t - \sqrt{\nu}k^2 t - \frac{1}{2}\alpha^2 \sqrt{\nu}t} e^{-a|x-z|} \\ &= e^{-\sqrt{\nu}k^2 t - \frac{1}{2}\alpha^2 \sqrt{\nu}t} e^{-\frac{|x-z|^2}{4\sqrt{\nu}t}}\end{aligned}$$

and recalling $i\alpha\varepsilon = \sqrt{\nu}$ and $\hat{\mu}_f = \nu^{-1/4} \sqrt{\lambda + i\alpha U + \alpha^2 \sqrt{\nu}}$, we have

$$\begin{aligned}\left| \frac{d\lambda}{\alpha\varepsilon \hat{\mu}_f(x)} \right| &= \left| \frac{d\lambda}{\nu^{1/4} \sqrt{(a + ik)^2 \sqrt{\nu} + i\alpha(U - \min_{[x,z]} U)}} \right| \\ &\leq \frac{2\sqrt{\nu}|a - ik|dk}{\sqrt{\nu}|a + ik|} \leq 2dk.\end{aligned}$$

Hence, we can estimate

$$\begin{aligned}\left| \int_{\Gamma_{\alpha,2}} e^{\lambda t} \mathcal{G}_a(x, z) \frac{d\lambda}{i\alpha} \right| &\leq 2 \int_{\mathbb{R}_+} e^{-\sqrt{\nu}k^2 t - \frac{1}{2}\alpha^2 \sqrt{\nu}t} e^{-\frac{|x-z|^2}{4\sqrt{\nu}t}} dk \\ &\leq C_0 (\sqrt{\nu}t)^{-1/2} e^{-\frac{1}{2}\alpha^2 \sqrt{\nu}t} e^{-\frac{|x-z|^2}{4\sqrt{\nu}t}}.\end{aligned}\tag{4.9}$$

Here, it is natural as the same for the heat kernel that the Green function has singularity in small time. To remove this singularity, it is necessary to take L^1 norm of the Green kernel as follows. By a view of the boundary layer behavior $\omega_\alpha(z)$ as in (4.5), the above yields

$$\begin{aligned}&\left| \int_0^\infty \int_{\Gamma_{\alpha,2}} e^{\lambda t} \mathcal{G}_a(x, z) \omega_\alpha(x) \frac{d\lambda}{i\alpha} dx \right| \\ &\leq C_0 e^{-\frac{1}{2}\alpha^2 \sqrt{\nu}t} \int_0^\infty (\sqrt{\nu}t)^{-1/2} e^{-\frac{|x-z|^2}{4\sqrt{\nu}t}} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) \right) e^{-\beta x} dx.\end{aligned}\tag{4.10}$$

In the case when $|x - z| \geq 8\beta\sqrt{\nu t}$, it is clear that

$$e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} e^{-\beta|x|} \leq e^{-\beta|z|} e^{-|x-z|\left(\frac{|x-z|}{8\sqrt{\nu t}} - \beta\right)} \leq e^{-\beta|z|}.$$

Whereas, for $|x - z| \leq 8\beta\sqrt{\nu t}$, we note that

$$e^{-\frac{1}{2}\sqrt{\nu t}} e^{-\beta|x|} \leq e^{-\frac{1}{16\beta}|x-z|} e^{-\beta|x|} \leq e^{-\beta|z|}$$

for $16\beta^2 \leq 1$. That is, the exponential decay $e^{-\beta z}$ is recovered after the integration. Precisely, this proves

$$e^{-\frac{1}{2}\alpha^2\sqrt{\nu t}} e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} e^{-\beta x} \leq e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu t}} e^{-\beta|z|}$$

in all the cases. It remains to study the integral

$$\int_0^\infty (\sqrt{\nu t})^{-1/2} e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}x)\right) dx. \quad (4.11)$$

First, without the boundary layer behavior, a straightforward computation gives

$$\int_0^\infty (\sqrt{\nu t})^{-1/2} e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} dx \leq C_0.$$

Next, we treat the boundary layer term. Since $\phi_{P-1+q}(\delta^{-1}x)$ is decreasing in x , we have

$$\begin{aligned} & \int_{z/2}^\infty (\sqrt{\nu t})^{-1/2} e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ & \leq C_0 \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \int_{z/2}^\infty (\sqrt{\nu t})^{-1/2} e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} dx \\ & \leq C_0 \delta^{-q} \phi_{P-1+q}(\delta^{-1}z). \end{aligned}$$

Whereas on $x \in (0, \frac{z}{2})$, we have $|x - z| \geq \frac{z}{2}$ and $\phi_{P-1+q} \leq 1$. Hence,

$$\int_0^{z/2} (\sqrt{\nu t})^{-1/2} e^{-\frac{|x-z|^2}{8\sqrt{\nu t}}} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \leq C_0 e^{-\frac{|z|^2}{32\sqrt{\nu t}}} \delta^{-q}.$$

We consider two cases. First, when $z \geq \theta_0 \nu^{1/4} t$, we use the exponential localized term $e^{-\frac{|z|^2}{32\sqrt{\nu t}}}$, yielding

$$e^{-\frac{|z|^2}{32\sqrt{\nu t}}} \leq e^{-\frac{\theta_0|z|}{32\nu^{1/4}}} \leq C_0 \phi_{P-1+q}(\delta^{-1}z),$$

in which the last inequality is due to the fact that $\delta = \nu^{1/4}$ and the boundary layer weight $\phi_{P-1+q}(\cdot)$ is a polynomial. Next, when $z \leq \theta_0 \nu^{1/4} t$, the boundary layer behavior is recovered from the time growing factor: precisely,

$$e^{-\theta_0 t} \leq e^{-\nu^{-1/4} z} \leq C_0 \phi_{P-1+q}(\delta^{-1} z).$$

This proves that

$$\int_0^\infty (\sqrt{\nu} t)^{-1/2} e^{-\frac{|x-z|^2}{8\sqrt{\nu} t}} \delta^{-q} \phi_{P-1+q}(\delta^{-1} x) dx \leq C_0 e^{\theta_0 t} \delta^{-q} \phi_{P-1+q}(\delta^{-1} z).$$

Summing up the above estimates in q from 1 to p into the integral (4.11), we obtain

$$\left\| \int_0^\infty \int_{\Gamma_{\alpha,2}} e^{\lambda t} \mathcal{G}_a(x, \cdot) \omega_\alpha(x) \frac{d\lambda}{i\alpha} dx \right\|_{\beta, \gamma, p} \leq C_0 e^{\theta_0 t} e^{-\frac{1}{2}(\alpha^2 - 1)\sqrt{\nu} t}. \quad (4.12)$$

Bounds on $\Gamma_{\alpha,4}$ and $\Gamma_{\alpha,5}$.

Finally, we give estimates on $\Gamma_{\alpha,4}$ and $\Gamma_{\alpha,5}$. Again by symmetry, it suffices to focus on $\Gamma_{\alpha,4}$. In this case, $a^2 \sqrt{\nu} \leq \theta_0$. We re-parametrize the contour $\Gamma_{\alpha,4}$ as follows:

$$\lambda = k - \alpha^2 \sqrt{\nu} - i\alpha \min_{[x,z]} U, \quad a^2 \sqrt{\nu} + \frac{1}{2} \alpha^2 \sqrt{\nu} \leq k \leq \theta_0 + \frac{1}{2} \alpha^2 \sqrt{\nu}.$$

We note

$$\hat{\mu}_f(y) = \nu^{-1/4} \sqrt{k + i\alpha(U - \min_{[x,z]} U)} \geq \nu^{-1/4} \sqrt{k}$$

and, recalling the definition of $a = \frac{|x-z|}{2\sqrt{\nu} t}$, we compute

$$\begin{aligned} \lambda t - \int_x^z \hat{\mu}_f(y) dy &\leq -\alpha^2 \sqrt{\nu} t + kt - \nu^{-1/4} \sqrt{k} |x - z| \\ &\leq -\frac{1}{2} \alpha^2 \sqrt{\nu} t + \theta_0 t - a |x - z| \\ &= -\frac{1}{2} \alpha^2 \sqrt{\nu} t + \theta_0 t - \frac{|x - z|^2}{2\sqrt{\nu} t}. \end{aligned}$$

Hence, we can estimate

$$\begin{aligned} \left| \int_{\Gamma_{\alpha,4}} e^{\lambda t} \mathcal{G}_a(x, z) \frac{d\lambda}{i\alpha} \right| &\leq \int_{a^2 \sqrt{\nu} + \frac{1}{2} \alpha^2 \sqrt{\nu}}^{\theta_0 + \frac{1}{2} \alpha^2 \sqrt{\nu}} e^{\theta_0 t - \frac{1}{2} \alpha^2 \sqrt{\nu} t} e^{-\frac{|x-z|^2}{2\sqrt{\nu} t}} dk \\ &\leq \theta_0 e^{\theta_0 t - \frac{1}{2} \alpha^2 \sqrt{\nu} t} e^{-\frac{|x-z|^2}{2\sqrt{\nu} t}}. \end{aligned}$$

Using the inequality $e^{-\frac{1}{2}\sqrt{\nu}t} \leq C_0(\sqrt{\nu}t)^{-1/2}$. The above yields

$$\left| \int_{\Gamma_{\alpha,4}} e^{\lambda t} \mathcal{G}_a(x, z) \frac{d\lambda}{i\alpha} \right| \leq \theta_0 e^{\theta_0 t - \frac{1}{2}(\alpha^2 - 1)\sqrt{\nu}t} (\sqrt{\nu}t)^{-1/2} e^{-\frac{|x-z|^2}{2\sqrt{\nu}t}}$$

which is the same as the bound on $\Gamma_{\alpha,2}$; see (4.9). This completes the proof of the bounds for \mathcal{S}_α as claimed in Proposition 2.3.

4.2 Bounds on \mathcal{R}_α .

We now prove the bounds on \mathcal{R}_α claimed in Propositions 2.3. Again, we need to choose the contour Γ_α in (2.6), which remains outside the complex strip

$$\mathcal{V}_\alpha := \left\{ \lambda = -k - \alpha^2 \sqrt{\nu} + i\alpha U(z), \quad k \in \mathbb{R}_+, \quad z \in \mathbb{R}_+ \right\}$$

across which the behavior of the Green function changes. Let M be a number so that $1 + \|U\|_{L^\infty} \leq M$, and let

$$\gamma = \Re \lambda_0 + \tau \tag{4.13}$$

for arbitrary, but fixed, constant $\tau > 0$. In what follows, we shall use the Cauchy's theory to decompose the contour Γ_α , appropriately. The contour Γ_α will be chosen so that it remains in the resolvent set of L_α and the normalized Evans function $D(\alpha, c)$ never vanishes; see Figure 2. Thus, there holds

$$|[D(\alpha, c)]^{-1}| \leq C_\tau$$

for some C_τ that depends on τ in (4.13). By Theorem 2.2, the residual Green function $\mathcal{R}_G(x, z)$ satisfies

$$\begin{aligned} |\mathcal{R}_G(x, z)| &\leq \frac{C_0}{|\varepsilon m_f^2|} \left(\frac{1}{\mu_s} e^{-\theta_0 \mu_s |x-z|} + \frac{1}{m_f} e^{-\theta_0 m_f |x-z|} \right) \\ &\quad + \frac{C_\tau}{|\varepsilon m_f^2|} \left(\frac{1}{\mu_s} e^{-\theta_0 \mu_s (|z|+|x|)} + \frac{1}{m_f} e^{-\theta_0 m_f (|z|+|x|)} \right). \end{aligned} \tag{4.14}$$

We estimate term by term by first integrating the convolution in x and then the λ -integral. Again, we assume $\|\omega_\alpha\|_{\beta, \gamma, p} = 1$. We have

$$\begin{aligned} &\int_0^\infty \mu_s^{-1} e^{-\theta_0 \mu_s |x-z|} |\omega_\alpha(x)| dx \\ &\leq \int_0^\infty \mu_s^{-1} e^{-\theta_0 \mu_s |x-z|} e^{-\beta x} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1} x) \right) dx. \end{aligned} \tag{4.15}$$

Using $\frac{1}{2}\theta_0\mu_s \geq \beta$ and the triangle inequality $|x| \geq |z| - |x - z|$, we obtain

$$e^{-\frac{1}{2}\theta_0\mu_s|x-z|}e^{-\beta x} \leq e^{-\beta|z|}$$

yielding the exponential decay in the boundary layer norm. We shall now estimate each term in (4.15). First, it is clear that

$$\int_0^\infty \mu_s^{-1} e^{-\frac{1}{2}\theta_0\mu_s|x-z|} dx \leq C_0\mu_s^{-2}.$$

As for the boundary layer term, we consider two cases. First, when $\mu_s\delta \leq 1$, we integrate the boundary layer term, yielding

$$\begin{aligned} & \int_0^\infty \mu_s^{-1} e^{-\frac{1}{2}\theta_0\mu_s|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ & \leq C_0\mu_s^{-1} \int_0^\infty \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ & \leq C_0\mu_s^{-1} \delta^{1-q}. \end{aligned}$$

In the above, we stress that we gain a factor δ due to the boundary layer behavior, with the thickness of order δ . In the case when $\mu_s\delta \geq 1$, the boundary layer behavior coming from the Green function has smaller thickness. Hence, using $\phi_{P-1+q} \leq 1$ and its decreasing property, we compute

$$\begin{aligned} & \int_0^\infty \mu_s^{-1} e^{-\frac{1}{2}\theta_0\mu_s|x-z|} \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) dx \\ & \leq C_0\mu_s^{-1} \delta^{-q} \left[e^{-\frac{1}{8}\theta_0\mu_s|z|} \int_0^{z/2} e^{-\frac{1}{4}\theta_0\mu_s|x-z|} dx \right. \\ & \quad \left. + \phi_{P-1+q}(\delta^{-1}z) \int_{z/2}^\infty e^{-\frac{1}{2}\theta_0\mu_s|x-z|} dx \right] \\ & \leq C_0\mu_s^{-2} \delta^{-q} \left[e^{-\frac{1}{8}\theta_0\mu_s|z|} + \phi_{P-1+q}(\delta^{-1}z) \right] \\ & \leq C_0\mu_s^{-2} \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \end{aligned} \tag{4.16}$$

in which the last inequality used the fact that $\mu_s\delta \geq 1$.

To summarize, this proves

$$\begin{aligned} & \int_0^\infty \mu_s^{-1} e^{-\theta_0\mu_s|x-z|} |\omega_\alpha(x)| dx \\ & \leq C_0 e^{-\beta z} \mu_s^{-2} \left(1 + \sum_{q=1}^p \delta^{-q} \left[\mu_s \delta \chi_{\{\mu_s\delta \leq 1\}} + \phi_{P-1+q}(\delta^{-1}z) \right] \right). \end{aligned} \tag{4.17}$$

Certainly, the above estimate holds for the other slow behavior in the Green function, namely the $e^{-\theta_0 \mu_s(|x|+|z|)}$ term, whose proof is identical to the above analysis and is therefore skipped.

Let us next consider the fast behavior in the Green function. We recall that

$$|\dot{\mu}_f^2| = \left| \alpha^2 + \frac{U-c}{\varepsilon} \right| \gtrsim \alpha^2 + \frac{1}{\sqrt{\nu}},$$

upon recalling that we consider the range of α and c so that $|\alpha(U-c)| \gtrsim 1$. In particular, $m_f \gtrsim \delta^{-1}$. That is, the boundary layer behavior of the fast mode dominates that of ω_α . Precisely, using $\phi_{P-1+q} \leq 1$, we have

$$\begin{aligned} & \int_0^\infty |m_f|^{-1} e^{-\theta_0 m_f(|x|+|z|)} |\omega_\alpha(x)| \, dx \\ & \leq \int_0^\infty |m_f|^{-1} e^{-\theta_0 m_f(|x|+|z|)} e^{-\beta x} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) \right) \, dx \\ & \leq C_0 e^{-\theta_0 m_f|z|} \int_0^\infty |m_f|^{-1} e^{-\theta_0 m_f|x|} e^{-\beta x} \left(1 + \delta^{-p} \right) \, dx \\ & \leq C_0 e^{-\theta_0 m_f|z|} |m_f|^{-2} (1 + \delta^{-p}) \\ & \leq C_0 e^{-\beta z} |m_f|^{-2} \left(1 + \delta^{-p} e^{-\theta_1 m_f z} \right) \\ & \leq e^{-\beta z} |m_f|^{-2} \left(1 + \delta^{-p} \phi_{P-1+p}(\delta^{-1}z) \right). \end{aligned}$$

Finally, we estimate

$$\begin{aligned} & \int_0^\infty |m_f|^{-1} e^{-\theta_0 m_f|x-z|} |\omega_\alpha(x)| \, dx \\ & \leq \int_0^\infty |m_f|^{-1} e^{-\theta_0 m_f|x-z|} e^{-\beta x} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}x) \right) \, dx. \end{aligned}$$

Splitting the integral into one over $(0, \frac{z}{2})$ and the other on $(\frac{z}{2}, \infty)$, exactly as done in (4.16), we obtain

$$\begin{aligned} & \int_0^\infty |m_f|^{-1} e^{-\theta_0 m_f|x-z|} |\omega_\alpha(x)| \, dx \\ & \leq C_0 e^{-\beta z} |m_f|^{-2} \left(1 + \sum_{q=1}^p \delta^{-q} \left[e^{-\frac{1}{2}\theta_0 m_f z} + \phi_{P-1+q}(\delta^{-1}z) \right] \right) \quad (4.18) \\ & \leq C_0 e^{-\beta z} |m_f|^{-2} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1}z) \right). \end{aligned}$$

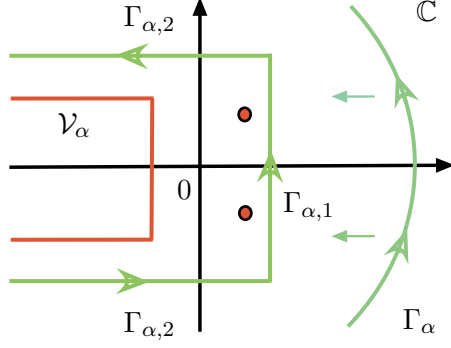


Figure 2: Shown the contour Γ_α of integration, when $\alpha t \gtrsim 1$.

To summarize, we have obtained the following convolution estimate

$$\begin{aligned}
 & \left| \int_0^\infty \mathcal{R}_G(x, z) \omega_\alpha(x) \frac{dx}{i\alpha} \right| \\
 & \leq C_0 |\alpha \varepsilon m_f^2|^{-1} \alpha^{-2} e^{-\beta z} \sum_{q=1}^p \left(1 + \delta^{-q} \phi_{P-1+q}(\delta^{-1} z) + \chi_{\{\alpha \delta \leq 1\}} \alpha \delta^{1-q} \right) \\
 & \quad + C_0 |\alpha \varepsilon m_f^2|^{-1} m_f^{-2} e^{-\beta z} \sum_{q=1}^p \left(1 + \delta^{-q} \phi_{P-1+q}(\delta^{-1} z) \right).
 \end{aligned} \tag{4.19}$$

In the above, we may use the fact that $m_f \gg \mu_s$ and $\mu_s = \alpha$ to simplify the bound. Using this, we can now integrate the above with respect to λ along the contour Γ_α of integration.

Case 1: $\alpha t \gtrsim 1$.

In this case, we take the contour Γ_α , for each $\alpha \in \mathbb{N}^*$, defined by

$$\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,2} \tag{4.20}$$

with

$$\Gamma_{\alpha,1} := [\gamma - i\alpha M, \gamma + i\alpha M], \quad \Gamma_{\alpha,2} := \left\{ \gamma - k \pm i\alpha M, \quad k \in \mathbb{R}_+ \right\}$$

in which $[\cdot, \cdot]$ denotes a segment in the complex plane; see Figure 2. We recall that

$$i\alpha \varepsilon m_f^2 = \lambda + i\alpha \inf_{\mathbb{R}_+} U + \alpha^2 \sqrt{\nu}.$$

We integrate

$$\begin{aligned}
\int_{\Gamma_{\alpha,1}} \frac{e^{\lambda t} |d\lambda|}{|\lambda + i\alpha \inf_{\mathbb{R}_+} U + \alpha^2 \sqrt{\nu}|} &\leq \int_{-M}^M \frac{e^{\gamma t} \alpha dk}{\sqrt{\gamma^2 + \alpha^2 (k + \inf_{\mathbb{R}_+} U)^2}} \\
&\leq \int_{-M+\inf_{\mathbb{R}_+} U}^{M+\inf_{\mathbb{R}_+} U} \frac{e^{\gamma t} dk}{\sqrt{\gamma^2 \alpha^{-2} + k^2}} \\
&\leq C_0 \int_{-M+\inf_{\mathbb{R}_+} U}^{M+\inf_{\mathbb{R}_+} U} \frac{e^{\gamma t} dk}{k + \gamma \alpha^{-1}} \\
&\leq C_0 e^{\gamma t} \log \alpha.
\end{aligned} \tag{4.21}$$

Here, we note that the above computation holds for all α and t (that is, including the case when $\alpha t \leq 1$). Similarly, we now compute the integral on $\Gamma_{\alpha,2}$. We have

$$\begin{aligned}
\int_{\Gamma_{\alpha,2}} \frac{e^{\lambda t} |d\lambda|}{|\lambda + i\alpha \inf_{\mathbb{R}_+} U + \alpha^2 \sqrt{\nu}|} &\leq C_0 \int_{\mathbb{R}_+} \frac{e^{\gamma t} e^{-kt} dk}{\sqrt{(\gamma - k)^2 + \frac{1}{4} \alpha^2 M^2}} \\
&\leq C_0 \int_{\mathbb{R}_+} \frac{e^{\gamma t} e^{-\ell} d\ell}{\sqrt{\ell^2 + \frac{1}{4} \alpha^2 M^2 t^2}} \\
&\leq C_0 e^{\gamma t},
\end{aligned}$$

in which the assumption $\alpha t \gtrsim 1$ was used.

Case 2: $\alpha t \ll 1$.

In this case, the previous computation yields a singularity in small time. We shall take the λ -integration first to avoid the singularity. We decompose the contour of integration as follows:

$$\begin{aligned}
\Gamma_{\alpha,1} &:= \{|\lambda| = \alpha M\} \cap \{\lambda \geq 0\} \\
\Gamma_{\alpha,2}^{\pm} &:= \left\{ \lambda = -|k| + ik \pm i\alpha M, \quad k \in \mathbb{R}_{\pm} \right\}.
\end{aligned}$$

See the left figure in Figure 3. Since M is taken sufficiently large, Γ_{α} remains in the resolvent set and the estimate (4.14) holds.

Case 2a: Fast behavior. We start with the fast behavior in the Green function:

$$\int_0^{\infty} \int_{\Gamma_{\alpha}} \frac{e^{\lambda t} e^{-\theta_0 m_f |x-z|} |\omega_{\alpha}(x)|}{\alpha |\varepsilon m_f^2| |m_f|} |d\lambda| dx. \tag{4.22}$$

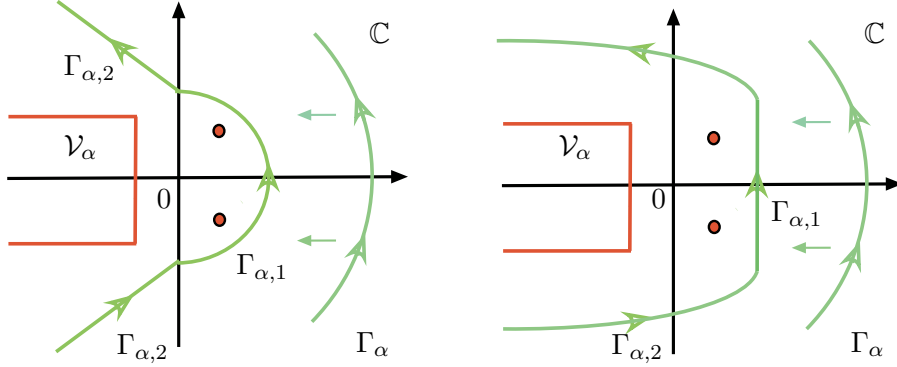


Figure 3: Shown the contour Γ_α of integration, when $\alpha t \ll 1$. Left is for the fast behavior, and right for the slow behavior in the residual Green function.

On $\Gamma_{\alpha,1}$, since $m_f \geq \sqrt{\alpha}\nu^{-1/4}$, we compute

$$\begin{aligned} \int_{\Gamma_{\alpha,1}} \frac{e^{\lambda t} e^{-\theta_0 m_f |x-z|}}{\alpha |\varepsilon m_f^2| |m_f|} |d\lambda| &\leq \int_{\{|\lambda|=\alpha M\}} \nu^{1/4} \alpha^{-3/2} e^{\alpha M t} e^{-\theta_0 \sqrt{\alpha} \nu^{-1/4} |x-z|} |d\lambda| \\ &\leq C_0 \nu^{1/4} \alpha^{-1/2} e^{-\theta_0 \sqrt{\alpha} \nu^{-1/4} |x-z|} \end{aligned}$$

upon recalling that we are in the case when $\alpha t \ll 1$. Now on $\Gamma_{\alpha,2}^\pm$, we note that

$$m_f \geq \nu^{-1/4} \sqrt{k + \alpha}$$

with $\lambda = -|k| + ik \pm i\alpha M$. Hence,

$$\begin{aligned} \int_{\Gamma_{\alpha,2}^\pm} \frac{e^{\lambda t} e^{-\theta_0 m_f |x-z|}}{\alpha |\varepsilon m_f^2| |m_f|} |d\lambda| &\leq C_0 \int_{\mathbb{R}} \nu^{1/4} (k + \alpha)^{-3/2} e^{-kt} e^{-\theta_0 \sqrt{\alpha} \nu^{-1/4} |x-z|} dk \\ &\leq C_0 \nu^{1/4} e^{-\theta_0 \sqrt{\alpha} \nu^{-1/4} |x-z|} \int_{\mathbb{R}} (k + \alpha)^{-3/2} dk \\ &\leq C_0 \nu^{1/4} e^{-\theta_0 \sqrt{\alpha} \nu^{-1/4} |x-z|}. \end{aligned}$$

We now follow exactly the convolution estimate (4.18), yielding

$$\begin{aligned} &\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\lambda t} e^{-\theta_0 m_f |x-z|} |\omega_\alpha(x)|}{\alpha |\varepsilon m_f^2| |m_f|} |d\lambda| dx \\ &\leq C_0 \int_0^\infty \nu^{1/4} e^{-\theta_0 \sqrt{\alpha} \nu^{-1/4} |x-z|} e^{-\beta x} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1} x) \right) dx \\ &\leq C_0 e^{-\beta z} \nu^{1/2} \alpha^{-1/2} \left(1 + \sum_{q=1}^p \delta^{-q} \phi_{P-1+q}(\delta^{-1} z) \right). \end{aligned}$$

A similar estimate holds for the fast behavior in the Green function involving the term $e^{-\theta_0 m_f(|x|+|z|)}$.

Case 2b: Slow behavior. We now turn to the slow behavior, treating the integral

$$\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\lambda t} e^{-\theta_0 \mu_s |x-z|} |\omega_\alpha(x)|}{\alpha |\varepsilon m_f^2| |\mu_s|} |d\lambda| dx. \quad (4.23)$$

In this case, since $|\varepsilon m_f^2|^{-1} \mu_s^{-1}$ is no longer integrable for large λ , we are obliged to use the fast behavior from the Green function $\mathcal{G}_a(x, z)$. Indeed, we recall that the residual Green function $\mathcal{R}_G(x, z)$ is defined by

$$\mathcal{R}_G(x, z) = \int_0^\infty \mathcal{G}_a(y, z) U''(y) G_{\alpha, c}(x, y) dy$$

and so the integral (4.23) was in fact the following integral

$$\int_0^\infty \int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\lambda t} e^{-m_f |y-z|} e^{-\theta_0 \mu_s |x-y|} e^{-\eta_0 |y|} |\omega_\alpha(x)|}{\alpha |\varepsilon m_f| |\mu_s|} |d\lambda| dy dx. \quad (4.24)$$

Let us now estimate (4.24). We take the following contour of integration

$$\Gamma_{\alpha, 1} = [-\alpha^2 \sqrt{\nu} + a^2 - i\alpha M, -\alpha^2 \sqrt{\nu} + a^2 + i\alpha M]$$

and

$$\Gamma_{\alpha, 2}^\pm = \left\{ \lambda = -\alpha^2 \sqrt{\nu} + a^2 - k^2 + 2aik \pm i\alpha M, \quad k \in \mathbb{R}_\pm \right\}$$

in which

$$a := \frac{|y-z|}{2\nu^{1/4}t} + \sqrt{\alpha M}.$$

See the right figure in Figure 3. We start with $\Gamma_{\alpha, 1}$. Similarly to the estimate (4.21), with $\lambda = -\alpha^2 \sqrt{\nu} + a^2 - i\alpha c$, we compute

$$\hat{\mu}_f = \nu^{-1/4} \sqrt{\lambda + i\alpha U + \alpha^2 \sqrt{\nu}} = \nu^{-1/4} \sqrt{a^2 + i\alpha(U - c)}$$

and hence

$$\hat{\mu}_f \geq \nu^{-1/4} a.$$

Using this, and the assumption that $\alpha t \ll 1$, we have

$$\begin{aligned}
\int_{\Gamma_{\alpha,1}} \frac{e^{\lambda t} e^{-m_f |y-z|}}{\alpha^2 |\varepsilon m_f|} |d\lambda| &\leq C_0 e^{a^2 t - \nu^{-1/4} a |y-z|} e^{-\alpha^2 \sqrt{\nu} t} \int_{-M}^M \frac{dc}{\nu^{1/4} |\sqrt{a^2 + i\alpha(U-c)}|} \\
&\leq C_0 e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t} e^{\alpha M t} \int_{-M}^M \frac{dc}{\nu^{1/4} a} \\
&\leq C_0 \alpha^{-1/2} \nu^{-1/4} e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t} \\
&\leq C_0 \alpha^{-1/2} t^{1/2} (\sqrt{\nu} t)^{-1/2} e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t} \\
&\leq C_0 \alpha^{-1} (\sqrt{\nu} t)^{-1/2} e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t}.
\end{aligned}$$

Let us turn to the integral on $\Gamma_{\alpha,2}^\pm$. We compute

$$\begin{aligned}
\hat{\mu}_f(x) &= \nu^{-1/4} \sqrt{a^2 - k^2 + 2aik \pm i\alpha M + i\alpha U(x)} \\
&\geq \nu^{-1/4} \sqrt{a^2 - k^2 + 2aik} = \nu^{-1/4} a
\end{aligned}$$

and so

$$\begin{aligned}
\lambda t - m_f |y-z| &\leq -\alpha^2 \sqrt{\nu} t + (a^2 - k^2)t - \nu^{-1/4} a |y-z| \\
&\leq -\alpha^2 \sqrt{\nu} t - k^2 t - \frac{|y-z|^2}{4\sqrt{\nu} t} + M\alpha t.
\end{aligned}$$

Recalling $\alpha t \ll 1$, we thus have

$$\begin{aligned}
\int_{\Gamma_{\alpha,2}^\pm} \frac{e^{\lambda t} e^{-m_f |y-z|}}{\alpha^2 |\varepsilon m_f|} |d\lambda| &\leq C_0 e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t} \int_{\mathbb{R}} \alpha^{-1} \nu^{-1/4} e^{-k^2 t} dk \\
&\leq C_0 \alpha^{-1} (\sqrt{\nu} t)^{-1/2} e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t}
\end{aligned}$$

which is the same bound as that of $\Gamma_{\alpha,1}$. Thus, we can estimate the y -integration in (4.24). The above yields

$$\begin{aligned}
&\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\lambda t} e^{-m_f |y-z|} e^{-\theta_0 \mu_s |x-y|} e^{-\eta_0 |y|}}{\alpha |\varepsilon m_f| |\mu_s|} |d\lambda| dy \\
&\leq C_0 \int_0^\infty \alpha^{-1} (\sqrt{\nu} t)^{-1/2} e^{-\frac{|y-z|^2}{4\sqrt{\nu} t}} e^{-\alpha^2 \sqrt{\nu} t} e^{-\theta_0 \mu_s |x-y|} dy.
\end{aligned}$$

We consider two cases. First, when $|y-z| \geq 8\theta_0 \mu_s \sqrt{\nu} t$, with $\theta_0 \leq \frac{1}{4}$, we have

$$e^{-\frac{|y-z|^2}{8\sqrt{\nu} t}} e^{-\theta_0 \mu_s |x-y|} \leq e^{-|y-z| \left(\frac{|y-z|}{8\sqrt{\nu} t} - \theta_0 \mu_s \right)} e^{-\theta_0 \mu_s |x-z|} \leq e^{-\theta_0 \mu_s |x-z|}.$$

Whereas when $|y - z| \leq 8\theta_0\mu_s\sqrt{\nu}t$, we bound

$$e^{-\frac{1}{2}\alpha^2\sqrt{\nu}t}e^{-\theta_0\mu_s|x-y|} \leq e^{-\frac{1}{2}\alpha^2\sqrt{\nu}t}e^{\theta_0^2\mu_s^2\sqrt{\nu}t}e^{-\theta_0\mu_s|x-z|} \leq e^{-\theta_0\mu_s|x-z|}$$

upon recalling that $\mu_s = \alpha$. Combining these estimates, we obtain

$$\begin{aligned} & \int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\lambda t} e^{-m_f|y-z|} e^{-\theta_0\mu_s|x-y|} e^{-\eta_0|y|}}{\alpha|\varepsilon m_f||\mu_s|} |d\lambda| dy \\ & \leq C_0 e^{-\theta_0\mu_s|x-z|} \int_0^\infty \mu_s^{-1} (\sqrt{\nu}t)^{-1/2} e^{-\frac{|y-z|^2}{8\sqrt{\nu}t}} e^{-\frac{1}{2}\alpha^2\sqrt{\nu}t} dy \\ & \leq C_0 \mu_s^{-1} e^{-\theta_0\mu_s|x-z|} e^{-\frac{1}{2}\alpha^2\sqrt{\nu}t}. \end{aligned}$$

Finally, the x -integration against the boundary layer behavior $\omega_\alpha(x)$ with the above kernel was already treated in (4.17). The above estimate also holds for the case when $e^{-\mu_s|y-z|}$ is replaced by $e^{\mu_s(|y|+|z|)}$. This completes the proof of the bounds on \mathcal{R}_α as claimed in Proposition 2.3.

4.3 Derivative bounds on \mathcal{S}_α

As for the derivative bounds, we note that differentiating the equation for $\mathcal{S}_\alpha(t)[\omega_\alpha] := \mathcal{S}_\alpha\omega_\alpha$ yields

$$(\partial_t + i\alpha U + \alpha^2\sqrt{\nu})\partial_z\mathcal{S}_\alpha(t)[\omega_\alpha] - \sqrt{\nu}\partial_z^2(\partial_z\mathcal{S}_\alpha(t)[\omega_\alpha]) = -i\alpha U'(z)\mathcal{S}_\alpha(t)[\omega_\alpha].$$

Here, we write $\mathcal{S}_\alpha(t)[\omega_\alpha]$ to indicate the time dependence of the semigroup. This gives the following Duhamel's formula:

$$\partial_z\mathcal{S}_\alpha(t)[\omega_\alpha] = \mathcal{S}_\alpha(t)[\partial_z\omega_\alpha] - i\alpha \int_0^t \mathcal{S}_\alpha(t-s)[U'\mathcal{S}_\alpha(s)[\omega_\alpha]] ds.$$

Now, applying the bounds on \mathcal{S}_α obtained from Proposition 2.3, for $\tau > 0$, we obtain at once

$$\|\mathcal{S}_\alpha(t)[\partial_z\omega_\alpha]\|_{\beta,\gamma,p+1} \leq C_\tau e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \|\partial_z\omega_\alpha\|_{\beta,\gamma,p+1}.$$

We apply again Proposition 2.3, for some positive $\tau_1 < \tau$, on $\mathcal{S}_\alpha(t-s)$. Thanks to the embedding estimate (3.1): $\|\omega\|_{\beta,\gamma,p+1} \leq \|\omega\|_{\beta,\gamma,p}$, we get

$$\begin{aligned} & \alpha \left\| \int_0^t \mathcal{S}_\alpha(t-s)[U'\mathcal{S}_\alpha(s)[\omega_\alpha]] ds \right\|_{\beta,\gamma,p+1} \\ & \leq \int_0^t C_{\tau_1} e^{\tau_1(t-s)} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}(t-s)} \|\alpha U'\mathcal{S}_\alpha(s)[\omega_\alpha]\|_{\beta,\gamma,p+1} ds \\ & \leq \int_0^t C_{\tau_1} e^{\tau_1(t-s)} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}(t-s)} C_\tau e^{\tau s} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}s} \|\alpha\omega_\alpha\|_{\beta,\gamma,p} ds \\ & \leq C_\tau C_{\tau_1} e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \|\alpha\omega_\alpha\|_{\beta,\gamma,p} \end{aligned}$$

in which the condition $\tau_1 < \tau$ was used. Certainly, we may take $\tau_1 = \frac{\tau}{2}$. This proves that

$$\begin{aligned} & \|\partial_z \mathcal{S}_\alpha(t)[\omega_\alpha]\|_{\beta,\gamma,p+1} \\ & \leq C_\tau e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \left[\|\partial_z \omega_\alpha\|_{\beta,\gamma,p+1} + \|\alpha \omega_\alpha\|_{\beta,\gamma,p} \right]. \end{aligned} \quad (4.25)$$

Next, by induction, we assume that

$$\|\partial_z^k \mathcal{S}_\alpha \omega_\alpha\|_{\beta,\gamma,p+k} \leq C_{\tau,k} e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \sum_{a+b \leq k} \|\alpha^a \partial_z^b \omega_\alpha\|_{\beta,\gamma,p+b}.$$

The Duhamel formula for $\partial_z^{k+1} \mathcal{S}_\alpha \omega_\alpha$ reads

$$\partial_z^{k+1} \mathcal{S}_\alpha(t)[\omega_\alpha] = \mathcal{S}_\alpha(t)[\partial_z^{k+1} \omega_\alpha] - i\alpha \int_0^t \mathcal{S}_\alpha(t-s) \left[[\partial_z^{k+1}, U] \mathcal{S}_\alpha(s)[\omega_\alpha] \right] ds$$

in which $[\partial_z^{k+1}, U]h = \partial_z^{k+1}(Uh) - U\partial_z^{k+1}h$. By the induction assumption and the fact that $\|\cdot\|_{\beta,\gamma,p}$ is decreasing in p , we have

$$\begin{aligned} \|[\partial_z^{k+1}, U] \mathcal{S}_\alpha(s)[\omega_\alpha]\|_{\beta,\gamma,p+k} & \leq C_k \sum_{j=0}^k \|\partial_z^j \mathcal{S}_\alpha(s)[\omega_\alpha]\|_{\beta,\gamma,p+j} \\ & \leq C_{\tau,k+1} e^{\tau s} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}s} \sum_{a+b \leq k} \|\alpha^a \partial_z^b \omega_\alpha\|_{\beta,\gamma,p+b} \end{aligned}$$

Now applying Proposition 2.3, for some positive $\tau_1 < \tau$, on $\mathcal{S}_\alpha(t-s)$, we obtain

$$\begin{aligned} & \alpha \left\| \int_0^t \mathcal{S}_\alpha(t-s) \left[[\partial_z^{k+1}, U] \mathcal{S}_\alpha(s)[\omega_\alpha] \right] ds \right\|_{\beta,\gamma,p+k+1} \\ & \leq \int_0^t C_{\tau_1} e^{\tau_1(t-s)} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}(t-s)} \|[\partial_z^{k+1}, U] \mathcal{S}_\alpha(s)[\omega_\alpha]\|_{\beta,\gamma,p+k} ds \\ & \leq \alpha \int_0^t C_{\tau_1} e^{\tau_1(t-s)} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}(t-s)} C_{\tau,k+1} e^{\tau s} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}s} \\ & \quad \times \sum_{a+b \leq k} \|\alpha^a \partial_z^b \omega_\alpha\|_{\beta,\gamma,p+b} ds \\ & \leq C_{\tau,k+1} C_{\tau_1} e^{\tau t} e^{-\frac{1}{2}(\alpha^2-1)\sqrt{\nu}t} \sum_{a+b \leq k} \|\alpha^{a+1} \partial_z^b \omega_\alpha\|_{\beta,\gamma,p+b}. \end{aligned}$$

This proves the bound for $\partial_z^{k+1} \mathcal{S}_\alpha \omega_\alpha$ as claimed in Proposition 2.4.

4.4 Derivative bounds on $e^{L_\alpha t}$

For each initial data ω_α , we introduce the vorticity and stream function through

$$\theta_\alpha = e^{L_\alpha t} \omega_\alpha, \quad \Delta_\alpha \psi_\alpha = \theta_\alpha.$$

Then, by construction, we have

$$\left(\partial_t + i\alpha U - \sqrt{\nu} \Delta_\alpha \right) \theta_\alpha - i\alpha U'' \psi_\alpha = 0$$

with initial data $\theta_\alpha|_{t=0} = \omega_\alpha$. Taking the derivative yields

$$\left(\partial_t + i\alpha U - \sqrt{\nu} \Delta_\alpha \right) \partial_z \theta_\alpha = i\alpha \partial_z (U'' \psi_\alpha) - i\alpha U' \theta_\alpha$$

with initial data $\partial_z \theta_\alpha|_{t=0} = \partial_z \omega_\alpha$. By definition of the semigroup \mathcal{S}_α , we obtain the following Duhamel's integral formula:

$$\partial_z \theta_\alpha = \mathcal{S}_\alpha(t) [\partial_z \omega_\alpha] + i\alpha \int_0^t \mathcal{S}_\alpha(t-s) \left[\partial_z (U'' \psi_\alpha(s)) - U' \theta_\alpha(s) \right] ds.$$

Here, applying Proposition 2.3 for $\theta_\alpha = e^{L_\alpha t} \omega_\alpha$ (see (2.11)), and the elliptic estimates in Proposition 3.3 for the Laplacian equation $\Delta_\alpha \psi_\alpha = \theta_\alpha$, we obtain

$$\|\alpha \psi_\alpha\|_{\beta, \gamma, 1} + \|\partial_z \psi_\alpha\|_{\beta, \gamma, 1} + \|\theta_\alpha\|_{\beta, \gamma, 1} \leq C_\tau e^{(\Re \lambda_0 + \tau)t} e^{-\frac{1}{2}(\alpha^2 - 1)\sqrt{\nu}t} \|\omega_\alpha\|_{\beta, \gamma, 1}.$$

Using this into the Duhamel's formula, and repeating the identical lines in the above proof of (4.25), we obtain at once

$$\|\partial_z \theta_\alpha\|_{\beta, \gamma, 2} \leq C_\tau e^{(\Re \lambda_0 + \tau)t} e^{-\frac{1}{2}(\alpha^2 - 1)\sqrt{\nu}t} \left[\|\partial_z \omega_\alpha\|_{\beta, \gamma, 2} + \|\alpha \omega_\alpha\|_{\beta, \gamma, 1} \right]. \quad (4.26)$$

Finally, we note that $\partial_z^k \theta_\alpha$ solves

$$\left(\partial_t + i\alpha U - \sqrt{\nu} \Delta_\alpha \right) \partial_z^k \theta_\alpha = i\alpha \partial_z^k (U'' \psi_\alpha) - i\alpha [\partial_z^k, U] \theta_\alpha.$$

Again using the Duhamel's principle for $\partial_z^k \theta_\alpha$ and following the exact same proof of bounds on $\partial_z^k \mathcal{S}_\alpha(t) [\omega_\alpha]$, done just above, we obtain bounds on the derivatives as claimed in Proposition 2.4.

5 Construction of an approximate solution

Let us now construct an approximate solution u_{app} , which solves Navier-Stokes equations, up to very small error terms. The construction is very classical ([6]), and so we will not completely detail it. First, we introduce the rescaled time and space variables

$$T = \frac{t}{\sqrt{\nu}}, \quad X = \frac{x}{\sqrt{\nu}}, \quad Z = \frac{z}{\sqrt{\nu}}.$$

We then construct an approximate solution that exhibits L^∞ instability. Our approximate solution is of the form

$$u_{\text{app}}(t, x, z) = U_{\text{bl}}(\sqrt{\nu}t, z) + \nu^p \sum_{j=0}^M \nu^{j/2} u_j(t, x, z). \quad (5.1)$$

We obtain the following theorem.

Theorem 5.1. *Assume that $U(z)$ is spectrally unstable to the Euler equations, with the maximal unstable eigenvalue λ_0 . Then, there are approximate solutions u_{app} of the form (5.1) to the Navier-Stokes equations in the following sense: for arbitrarily large numbers s, p, M , the functions u_{app} approximately solve the nonlinear Navier-Stokes equations*

$$\begin{aligned} \partial_t u_{\text{app}} + (u_{\text{app}} \cdot \nabla) u_{\text{app}} + \nabla p_{\text{app}} &= \sqrt{\nu} \Delta u_{\text{app}} + \mathcal{E}_{\text{app}}, \\ \nabla \cdot u_{\text{app}} &= 0, \end{aligned} \quad (5.2)$$

for some remainder \mathcal{E}_{app} and for time $t \leq T_\nu$, with T_ν being defined through

$$\nu^p e^{\Re \lambda_0 T_\nu} = 1.$$

In addition, for all $t \in [0, T_\nu]$, there hold

$$\|\nabla \times u_j(t)\|_{H_{\text{bl}}^s} \leq C_{j,s} e^{(1+\frac{j}{2p})\Re \lambda_0 t}$$

$$\|\nabla \times \mathcal{E}_{\text{app}}(t)\|_{H_{\text{bl}}^s} \leq C_{M,s} \left(\nu^p e^{\Re \lambda_0 t} \right)^{1+\frac{M+1}{2p}}.$$

Here, $\|\cdot\|_{H_{\text{bl}}^s}$ denotes the boundary layer Sobolev norms defined as in (1.5).

Furthermore, there are positive constants $\theta_0, \theta_1, \theta_2$ independent of ν so that there holds

$$\theta_1 \nu^p e^{\Re \lambda_0 t} \leq \|(u_{\text{app}} - U_{\text{bl}})(t)\|_{L^\infty} \leq \theta_2 \nu^p e^{\Re \lambda_0 t}$$

as long as $\nu^p e^{\Re \lambda_0 t}$ remains smaller than θ_0 .

5.1 Formal construction

Let $v = u - U_{\text{bl}}$, where u denotes the genuine solution to the Navier-Stokes equations. Then, the vorticity $\omega = \nabla \times v$ solves

$$\partial_t \omega + (U_{\text{bl}}(\sqrt{\nu}t, y) + v) \cdot \nabla \omega + v_2 \partial_y^2 U_s(\sqrt{\nu}t, y) - \sqrt{\nu} \Delta \omega = 0$$

in which $v = \nabla^\perp \Delta^{-1} \omega$ and v_2 denotes the vertical component of velocity. Here and in what follows, Δ^{-1} is computed with the zero Dirichlet boundary condition. As U_{bl} depends slowly on time, we can rewrite the vorticity equation as follows:

$$(\partial_t - L)\omega + \sqrt{\nu} S\omega + Q(\omega, \omega) = 0. \quad (5.3)$$

In (5.3), L denotes the linearized Navier-Stokes operator around the stationary boundary layer $U = U_s(0, z)$:

$$L\omega := \sqrt{\nu} \Delta \omega - U \partial_x \omega - u_2 U'',$$

$Q(\omega, \tilde{\omega})$ denotes the quadratic nonlinear term $u \cdot \nabla \tilde{\omega}$, with $v = \nabla^\perp \Delta^{-1} \omega$, and S denotes the perturbed operator defined by

$$S\omega := \nu^{-1/2} [U_s(\sqrt{\nu}t, z) - U(z)] \partial_x \omega + \nu^{-1/2} u_2 [\partial_y^2 U_s(\sqrt{\nu}t, z) - U''(z)].$$

We shall construct approximate solutions to the vorticity equation (5.3) in the form

$$\omega_{\text{app}} = \nu^p \sum_{j=0}^M \nu^{j/2} \omega_j.$$

For sake of simplicity, we take p to be a (sufficiently large) integer. Plugging this Ansatz into (5.3) and matching order in ν , we are led to solve

- for $j = 0$:

$$(\partial_t - L)\omega_0 = 0$$

- for $0 < j \leq M$:

$$(\partial_t - L)\omega_j = R_j, \quad \omega_j|_{t=0} = 0, \quad (5.4)$$

in which the remainders R_j is defined by

$$R_j = S\omega_{j-1} + \sum_{k+\ell+2p=j} Q(\omega_k, \omega_\ell).$$

As a consequence, the error of the approximation is then

$$R^{\text{app}} = \nu^{p+\frac{M+1}{2}} S\omega_M + \sum_{k+\ell > M+1-2p; 1 \leq k, \ell \leq M} \nu^{2p+\frac{k+\ell}{2}} Q(\omega_k, \omega_\ell) \quad (5.5)$$

which formally is of order $\nu^{p+\frac{M+1}{2}}$, for arbitrary p and M .

5.2 Growing mode

We take ω_0 to be the maximal growing mode of $\partial_t - L$. By construction,

$$\omega_0 = e^{\lambda_\nu t} \Delta(e^{i\alpha_\nu x} \phi_0(z)) + \text{complex conjugate} \quad (5.6)$$

with

$$\phi_0 := \phi_{in,0}(z) + \delta_{bl} \phi_{bl,0}\left(\frac{z}{\delta_{bl}}\right)$$

solving the Orr-Sommerfeld problem and $\delta_{bl} = \nu^{1/4}$ being the thickness of boundary sublayers. Directly from the definition and the fact that $\phi_{bl,0}$ rapidly decays at infinity, it follows that

$$c_0 e^{\Re \lambda_\nu t} \leq \|\omega_0\|_{H_{bl}^{s+M}} \leq C_0 e^{\Re \lambda_\nu t}$$

for some positive constants c_0, C_0 . In addition, the unstable eigenvalue λ_ν satisfies

$$\lambda_\nu = \lambda_0 + \mathcal{O}(\sqrt{\nu})$$

in the inviscid limit, with λ_0 being the maximal unstable eigenvalue of the linearized Euler equations around U . This proves that the corresponding vorticity ω_0 defined as in (5.6) satisfies

$$c_0 e^{\Re \lambda_0 t} \leq \|\omega_0\|_{H_{bl}^{s+M}} \leq C_0 e^{\Re \lambda_0 t} \quad (5.7)$$

as long as $\sqrt{\nu}t$ remains bounded.

5.3 Higher order profiles

Let us solve (5.4) for ω_j . We shall prove by induction that

$$\|\omega_j\|_{H_{bl}^{s+M+1-j}} \leq C_{j,s} e^{(1+\frac{j}{2p})\Re \lambda_0 t} \quad (5.8)$$

for all $j \geq 0$. This in particular yields the claimed H_{bl}^s estimates for ω_j for $0 \leq j \leq M$. The case $j = 0$ is proved in (5.7). We assume that (5.8) holds for $j \geq 0$, and we shall prove it for $j + 1$.

First, since U_s solves the heat equation with initial data $U(z)$, we have

$$|U_s(\sqrt{\nu}t, z) - U(z)| \leq C \|U''\|_{L^\infty} \sqrt{\nu}t$$

and

$$|\partial_y^2 U_s(\sqrt{\nu}t, z) - U''(z)| \leq C \|U''\|_{W^{2,\infty}} \sqrt{\nu}t.$$

Similar bounds hold for z -derivatives. Hence, by definition and Lemma 3.5, we compute

$$\begin{aligned}\|S\omega(t)\|_{H_{\text{bl}}^{s'}} &\leq \nu^{-1/2} \| [U_s(\sqrt{\nu}t) - U] \partial_x \omega \|_{H_{\text{bl}}^{s'}} + \nu^{-1/2} \| u_2 [\partial_y^2 U_s(\sqrt{\nu}t) - U''] \|_{H_{\text{bl}}^{s'}} \\ &\leq \nu^{-1/2} \| [U_s(\sqrt{\nu}t) - U] \|_{H_{\text{bl}}^{s'+1}} \| \omega \|_{H_{\text{bl}}^{s'+1}} \\ &\leq Ct \| \omega \|_{H_{\text{bl}}^{s'+1}}\end{aligned}$$

for all $s' > 1$.

By using again Lemma 3.5 to the quadratic nonlinear term $Q(\omega_k, \omega_\ell) = u_k \cdot \nabla \omega_j$ and the induction hypothesis (5.8), the remainder is estimated by

$$\begin{aligned}\|R_{j+1}(t)\|_{H_{\text{bl}}^{s+M-j}} &\leq \|S\omega_j\|_{H_{\text{bl}}^{s+M-j}} + \sum_{k+\ell+2p=j+1} \|Q(\omega_k, \omega_\ell)\|_{H_{\text{bl}}^{s+M-j}} \\ &\lesssim t \|\omega_j\|_{H_{\text{bl}}^{s+M+1-j}} + \sum_{k+\ell+2p=j+1} \|\omega_k\|_{H_{\text{bl}}^{s+M+1-j}} \|\omega_\ell\|_{H_{\text{bl}}^{s+M+1-j}} \\ &\lesssim t \|\omega_j\|_{H_{\text{bl}}^{s+M+1-j}} + \sum_{k+\ell+2p=j+1} \|\omega_k\|_{H_{\text{bl}}^{s+M+1-k}} \|\omega_\ell\|_{H_{\text{bl}}^{s+M+1-\ell}} \\ &\lesssim t e^{(1+\frac{j}{2p})\Re\lambda_0 t} + \sum_{k+\ell+2p=j+1} e^{(1+\frac{k}{2p})\Re\lambda_0 t} e^{(1+\frac{\ell}{2p})\Re\lambda_0 t} \\ &\lesssim e^{(1+\frac{j+1}{2p})\Re\lambda_0 t}.\end{aligned}$$

Finally, by a view of (5.4), the Duhamel principle and the semigroup bound in H_{bl}^s Sobolev spaces obtained in Theorem 1.1 yield

$$\begin{aligned}\|\omega_{j+1}\|_{H_{\text{bl}}^{s+M-j}} &\leq \int_0^t \|e^{L(t-\tau)} R_{j+1}(\tau)\|_{H_{\text{bl}}^{s+M-j}} d\tau \\ &\leq C_\beta \int_0^t e^{(\Re\lambda_0 + \beta)(t-\tau)} \|R_{j+1}(\tau)\|_{H_{\text{bl}}^{s+M-j}} d\tau \\ &\leq C_\beta \int_0^t e^{(\Re\lambda_0 + \beta)(t-\tau)} e^{(1+\frac{j+1}{2p})\Re\lambda_0 \tau} d\tau.\end{aligned}$$

By taking β small so that $\beta < \frac{j+1}{2p}$, the above integral is bounded by $C_\beta e^{(1+\frac{j+1}{2p})\Re\lambda_0 t}$, which completes the proof of (5.8) for all $j \geq 0$.

5.4 Remainder

It remains to bound the error term R^{app} defined as in (5.5). We estimate

$$\begin{aligned}
\|R^{\text{app}}\|_{H_{\text{bl}}^s} &\leq \nu^{p+\frac{M+1}{2}} \|S\omega_M\|_{H_{\text{bl}}^s} + \sum_{k+\ell+2p>M+1; 1\leq k,\ell\leq M} \nu^{2p+\frac{k+\ell}{2}} \|Q(\omega_k, \omega_\ell)\|_{H_{\text{bl}}^s} \\
&\lesssim \nu^{p+\frac{M+1}{2}} t e^{(1+\frac{M}{2p})\Re\lambda_0 t} \\
&\quad + \sum_{k+\ell+2p>M+1; 1\leq k,\ell\leq M} \nu^{2p+\frac{k+\ell}{2}} e^{(1+\frac{k}{2p})\Re\lambda_0 t} e^{(1+\frac{\ell}{2p})\Re\lambda_0 t} \\
&\lesssim \sum_{k=M+1}^{2M} \left(\nu^p e^{\Re\lambda_0 t} \right)^{1+\frac{k}{2p}}.
\end{aligned}$$

In particular, as long as $\nu^p e^{\Re\lambda_0 t}$ remains bounded, we obtain

$$\|\Phi(\omega_{\text{app}})\|_{H_{\text{bl}}^s} \lesssim \left(\nu^p e^{\Re\lambda_0 t} \right)^{1+\frac{M+1}{2p}}.$$

5.5 Proof of Theorem 5.1

The theorem follows by collecting all the previous bounds and using the elliptic estimates for the velocity in term of vorticity function.

5.6 Proof of Theorem 1.2

The theorem is a consequence of Theorem 5.1 on the approximate solutions. Indeed, let us define T_ν so that

$$\nu^p e^{\Re\lambda_0 T_\nu} = \nu^\delta.$$

Thus, for $t \in [0, T_\nu]$, Section 5.4 yields

$$\|\Phi(\omega_{\text{app}})\|_{H_{\text{bl}}^s} \lesssim \left(\nu^p e^{\Re\lambda_0 t} \right)^{1+\frac{M+1}{2p}} \lesssim \nu^{\delta(1+\frac{M+1}{2p})}.$$

By taking M sufficiently large, this error term or so-called sources in the Navier-Stokes equations (5.2), remains arbitrarily small of order ν^N for large N . When $\nu^p e^{\Re\lambda_0 t}$ reaches order one, the instability of order one in sup norm follows, leaving the error term now also of order one. This completes the proof of Theorem 1.2.

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